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# ORBITAL BEHAVIOR OF EARTH SATELLITES

BY

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Part II*

## 3. ORBITS ABOUT AN OBLATE SPHEROID

### *3.1 Introduction*

Some properties of orbits in the field of an oblate spheroid are mentioned by Brown (3)[2] and by standard texts on celestial mechanics (for example, (9) paragraph 147). The most extensive analyses of the problem seem to be those of MacMillan (8) and Brouwer (2). Brouwer remarks that the problem has received limited treatment probably because it is not widely applicable in the solar system. However, for a close, small earth satellite the oblateness effect may be expected to dominate most other perturbations.

The principal objections to using these treatments are that they are limited to small orbital inclinations, they choose basic variables which are not the most convenient for an easy physical visualization of the orbit, they usually focus on time behavior rather than orbital geometry (which often is of first importance in satellite engineering), and they employ rationales and arguments which frequently are very obscure to anyone without a substantial background in classical celestial mechanics. In short, their natural preoccupation with problems which are of primary interest for the analysis of natural satellites severely limits their usefulness to the non-astronomer who is interested in artificial satellites.

The effect of the earth's oblateness recently has been treated (1) with artificial satellites primarily in mind. Specifically, the motivating problem was the inference of oblateness from observation of the orbital

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[2] The boldface numbers in parentheses refer to the references appended to Part I of the paper.

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motion. Although there is no discussion of the relation between the method used and the existing classical approaches, and the analysis appears to suffer some of the disadvantages of these approaches in its choice of variables and the unwieldy form of its results, the case of large inclination is explicitly encompassed. Unfortunately, though, a circular rather than an elliptic generating solution is used. This seems unrealistic, for the early satellite the authors apparently had in mind ignores the well-known qualitative effect of the precession of the perigee and makes the results relatively trivial. Finally, as the authors themselves recognize, a first-order perturbation probably does not suffice in solving their motivating geophysical problem.[3]

The effect of oblateness on satellite motion is treated here *ab initio*. With the present approach, the form of the results is more directly applicable to the engineering and operational problems related to an artificial satellite. Although it does differ somewhat from the approach used in the astronomical literature, the differences are not too hard to bridge. The following comments may help put the method into proper perspective with the classical methods.[4]

The quantity $r^2\dot{\beta}$ is used as a dependent variable, as described below. It is the areal velocity constant of the two-body problem and commonly is taken as a basic parameter in the method of variation of parameters, together with four other similar two-body constants; the mean anomaly, a two-body variable, ordinarily being chosen as the sixth variable. The present approach represents the other extreme in which one two-body constant is used, together with five two-body variables.

Dr. Herrick points out that the use of the true anomaly $\beta$ as an independent variable parallels in part Hansen's perturbation theory, inasmuch as "such a choice inevitably throws a perturbation into the time." The advantage of this choice is heightened by choosing $u = 1/r$ as another dependent variable, for $u$ is a closed expression in the mean anomaly inclusive of two-body terms whereas both $u$ and $r$ are infinite series in the time. He observes that the same advantage exists when eccentric anomaly $E$ is used as the independent variable; indeed, that this choice may be preferable for small eccentricities in view of the simple representation in rectangular coordinates in the orbital plane for this case. However, some advantage would be lost thereby because the potential function and acceleration functions depend on inverse powers of $r$ which must be expanded in power series in $E$.

He comments further that:

> . . . frequent reliance on $\beta = \beta^* + \beta_P$ (astronomical notation $u = v + w$), where $\beta_P = w$ is the "argument of perigee" is a tacit

[3] Some of the deficiencies mentioned here have been remedied in subsequent papers by those authors.

[4] I am indebted to Dr. Samuel Herrick, Professor of Astronomy at UCLA, for most of these comments.

> recognition not only of the geometrical advantages of $\beta$ but also of the fact that it is much less subject to perturbational variations than $\beta = v$ when the eccentricity is very small. In this circumstance, the perturbational librations become enormous, and there is serious question as to whether $\beta^*$ would be a usable independent variable when the perturbations in the time are required. Then $\beta$ itself might prove to be the best available independent variable.

He properly notes that the primary purpose of the method is to provide a complete qualitative and approximate quantitative description of the motion, and that there remain significant problems in the prediction or interpolation of very accurate positions which require a careful study of higher order terms. For extreme accuracy, the number of terms in the series expansions to which the present method leads becomes too large to handle conveniently.

A theory of the motion which is completely adequate by conventional astronomical standards probably will be very difficult to construct. For high accuracy, it is not prudent to expect the present theory to serve for more than a few revolutions. However, Dr. Herrick concludes that two considerations indicate that the method will give an accuracy adequate for many satellite engineering purposes

> . . . if the mean angular motion or "orbital frequency" is determined by observations extending over a number of revolutions. One of these considerations is the absorption of the secular terms of $\beta$ into the mean angular motion. By this device the coefficient of the secular term of $\beta$, which is insufficiently determined by the first-order theory in view of the steady and indefinite increase of $\beta^*$ is observationally determined as a part of the mean angular motion. Two secular terms of this character remain: the one in $\beta_P$ that determines the position of perigee, and the one in $\Lambda_\Omega$ that determines the position of the node. The first of these fortunately is rendered of small effect by the facts that its coefficient involves the small eccentricity of the orbit and further affects only secondary terms in $r$ and $\beta$. The second has in its coefficient the factor $\cos\gamma$, which [may be small]; the uncertainty thereof is made small at the same time.

The specific goals of this presentation are

1. to deduce the orbit geometry in the field of an oblate spheroid, as if no other external forces act;
2. to seek a rotating reference frame with respect to which the orbit is periodic;
3. to relate the elements of the orbit to the kinematic and geometric initial conditions which exist when the orbit is established; and
4. to develop the time behavior of the basic dependent variables which describe the orbit.

The principal limitation of the analysis is its restriction to first order effects in the oblateness parameter. A subsequent analysis considering higher order terms is desirable, but I conjecture that the effects of these terms will be comparable to those of some of the smaller perturbation sources discussed in the previous chapter. Moreover, I feel that it would be inconsistent to continue to ignore the latter and the nonlinear cross-coupling effects while extending the existing oblateness analysis.

It is assumed at the outset that the orbital plane rotates at a constant angular velocity $\dot{\lambda}_0$ about the $Y$ axis. This angular rate is yet unspecified, but will be chosen from periodicity considerations. It will appear later that no more elaborate assumptions about the motion of the orbital plane are needed. In terms of coordinates $r, \alpha, \beta$ motions are sought which

1. are small and periodic out of the orbital plane, the inclination $\gamma$ of this plane remaining constant; and
2. have a projection on the orbital plane which is periodic and almost elliptical.

As the analysis develops, it is seen that these conditions cannot be satisfied even when the rotational angular velocity of the orbital plane can be freely chosen. One must also assume that the perigee of the approximating ellipse rotates in the orbital plane. The angular speed $\dot{\beta}_P$ of the rotation of the perigee is to be chosen with $\dot{\lambda}_0$ from periodicity considerations.

### *3.2 Basic Equations*

The equations of motion involve the potential function for the satellite, assumed[5] to be a point mass $m$, in the field of the earth, assumed to be an oblate spheroid rotationally symmetric about the $Y$ axis. This function is supposed to be of the form

$$V = -\frac{mK}{r}\left[1 + \mu\frac{a^2}{r^2}(1 - 3\sin^2\lambda_S) + \mu_2\frac{a^4}{r^4}(3 - 30\sin^2\lambda_S + 35\sin^4\lambda_S) + \cdots\right]. \quad (42)$$

Here $K$ is the product of the mass of the earth and the gravitational constant, $a$ is the equatorial radius of the earth, and $\mu$ and $\mu_2$ are dimensionless form factors of the earth.[6]

[5] The distribution of satellite mass is quite important in considering its attitude motion in the field of the earth, but the point mass assumption is excellent in treating the orbit.

[6] This conforms to Brouwer's point of view (2) that the coefficients in this expansion should be considered to be empirical constants of the earth.

In subsequent numerical work, the values adopted[7] for these parameters as a basis for calculation are

$$K = 3.986277 \times 10^{14} \text{ meter}^3 \text{ sec}^{-2}$$
$$a = 6.378388 \times 10^{6} \text{ meter}$$
$$\mu = 5.461 \times 10^{-4}.$$

For a homogeneous oblate ellipsoid (minor radius $b$), the form factors $\mu$ and $\mu_2$ have the interpretation $\mu = \frac{1}{10}(1-b^2/a^2)$, $\mu_2 = \frac{3}{280}(1-b^2/a^2)^2$. Therefore, although $\mu_2$ is somewhat larger for the actual earth, it can be expected to be roughly of the order of $\mu^2$. In the sequel, $\mu_2$-terms are neglected with $\mu^2$-terms.

The equations of motion can be written directly from Eq. 42 and an expression for the kinetic energy of the satellite in terms of $r$, $\alpha$, $\beta$, their time derivatives, and $\dot{\Lambda}_0$, namely

$$T = \frac{m}{2}\left[\dot{r}^2 + r^2(\dot{\alpha} - \dot{\Lambda}_0 \cos\beta \sin\gamma)^2 + r^2(\dot{\beta}\cos\alpha + \dot{\Lambda}_0 \cos\lambda_S \cos\zeta)^2\right]. \quad (43)$$

The equations are

$$\frac{d^2r}{dt^2} = r\left[(\dot{\alpha} - \dot{\Lambda}_0 \cos\beta \sin\gamma)^2 + (\dot{\beta}\cos\alpha + \dot{\Lambda}_0 \cos\lambda_S \cos\zeta)^2\right] - \frac{K}{r^2}\left[1 + 3\mu\frac{a^2}{r^2}(1 - 3\sin^2\lambda_S)\right] + 0(\mu^2,\mu_2) \quad (44)$$

$$\frac{d}{dt}\left[r^2(\dot{\alpha} - \dot{\Lambda}_0 \cos\beta \sin\gamma)\right] = -r^2\left[\dot{\beta}^2 \sin\alpha\cos\alpha + \dot{\Lambda}_0^2 \sin\lambda_S \times \cos\lambda_S \cos\zeta + \dot{\beta}\dot{\Lambda}_0(\cos\alpha \sin\lambda_S + \sin\alpha\cos\lambda_S\cos\zeta)\right] - \frac{6\mu K a^2}{r^3}\sin\lambda_S\cos\lambda_S\cos\zeta + 0(\mu^2,\mu_2) \quad (45)$$

$$\frac{d}{dt}\left[r^2\cos\alpha(\dot{\beta}\cos\alpha + \dot{\Lambda}_0\cos\lambda_S\cos\zeta)\right] = r^2\left[\alpha\dot{\Lambda}_0\sin\gamma\sin\beta - \dot{\beta}\dot{\Lambda}_0\sin\alpha\cos\alpha\sin\gamma\cos\beta - \dot{\Lambda}_0^2\cos\beta\sin\gamma\cos\alpha\sin\lambda_S\right] - \frac{6\mu K a^2}{r^3}\sin\lambda_S\cos\alpha\sin\gamma\cos\beta + 0(\mu^2,\mu_2). \quad (46)$$

However, the geometry rather than the time history of the orbit frequently is of principal concern, and it is natural to use an angle rather than time as the basic independent variable. Specifically, choosing

[7] These values have been recommended to me by Dr. Herrick.

$\beta^*$, the angular advance from the perigee, as the independent variable both focuses attention on the geometry where it is desired and simplifies the analysis significantly when periodicity is sought in this angle.

Introducing new dependent variables

$$u = r^{-1}; \quad P = r^2\dot{\beta}^* \tag{47}$$

and denoting derivatives with respect to $\beta^*$ by primes, the equations of motion become

$$(Pu')' + Pu[(\alpha' - \Lambda_0' \cos\beta \sin\gamma)^2 + (\cos\alpha + \beta_P' \cos\alpha + \Lambda_0' \cos\lambda_S \cos\zeta)^2] - \frac{K}{P}[1 + 3\mu a^2u^2(1 - 3\sin^2\lambda_S)] = 0 \tag{48}$$

$$[P(\alpha' - \Lambda_0' \cos\beta \sin\gamma)]' + P[(1 + \beta_P')^2 \sin\alpha \cos\alpha + \Lambda_0'^2 \sin\lambda_S \times \cos\lambda_S \cos\zeta + (1 + \beta_P')\Lambda_0' (\cos\alpha \sin\lambda_S + \sin\alpha \cos\lambda_S \times \cos\zeta)] + \frac{6\mu K a^2 u}{P} \sin\lambda_S \cos\lambda_S \cos\zeta = 0 \tag{49}$$

$$[P\cos\alpha(\cos\alpha + \beta_P' \cos\alpha + \Lambda_0' \cos\lambda_S \cos\zeta)] + P[-\alpha'\Lambda_0' \sin\gamma \times \sin\beta + (1 + \beta_P')\Lambda_0' \sin\alpha \cos\alpha \sin\gamma \cos\beta + \Lambda_0'^2 \cos\beta \times \sin\gamma \cos\alpha \sin\lambda_S] + \frac{6\mu K a^2 u}{P} \sin\lambda_S \cos\alpha \sin\gamma \cos\beta = 0 \tag{50}$$

in which it is to be remembered that $\beta - \beta_P = \beta^*$. In these equations, and in general hereafter, it is implicit that terms of the order $\mu^2$ and $\mu_2$ have been discarded.

### 3.3 *Perturbation Solution*

The general equations can be solved by a perturbation method. Suppose that the dependent variables and basic parameters can be expanded into power series in $\mu$, $\mu_2, \cdots$ of the form

$$u(\beta^*) = u_0(\beta^*) + \mu u_1(\beta^*) + \mu^2 u_2(\beta^*) + 0(\mu^3, \mu_2, \cdots) \tag{51}$$

$$P(\beta^*) = P_0(\beta^*) + \mu P_1(\beta^*) + \mu^2 P_2(\beta^*) + \cdots \tag{52}$$

$$\alpha(\beta^*) = \mu\alpha_1(\beta^*) + \mu^2\alpha_2(\beta^*) + \cdots \tag{53}$$

$$\Lambda_0' = \mu\Lambda_{01}' + \mu^2\Lambda_{02}' + \cdots \tag{54}$$

$$\gamma = \gamma_0 + \mu\gamma_1 + \mu^2\gamma_2 + \cdots \tag{55}$$

$$\beta_P = (\beta_0 + \mu\beta_1 + \mu^2\beta_2 + \cdots + \beta^*(\mu b_1 + \mu^2 b_2 + \cdots). \tag{56}$$

There is no loss in generality in beginning the expansions of some of these quantities with first-order terms in $\mu$, inasmuch as $\mu = 0$ corresponds to purely elliptic motion in which these quantities do not appear. Note that Eq. 56 implies that the perigee rotates slowly in the orbital plane as the satellite advances in its orbit.

Substituting Eqs. 51 to 56 into Eqs. 48 to 50 and equating to zero the coefficients of each power of $\mu$, one finds the perturbation equations. For the zero order,

$$P_0' = 0; \quad (P_0u_0')' + P_0u_0 = \frac{K}{P_0} \tag{57}$$

which implies immediately that $P_0$ is constant and that

$$u_0(\beta^*) = \frac{K}{P_0^2}(1 + \epsilon \cos \beta^*). \tag{58}$$

The constants of integration which arise in solving Eq. 57 are so chosen that the zero approximation is an ellipse with eccentricity $\epsilon$ and perigree at $\beta^* = 0 \bmod (2\pi)$.

The equations of the first approximation are

$$\alpha_1'' + \alpha_1 = -2\Lambda_{Q1}' \sin \gamma_0 \sin (\beta^* + \beta_0) - \frac{6Ka^2u_0}{P_0^3} \sin \gamma_0 \cos \gamma_0 \sin (\beta^* + \beta_0) \tag{59}$$

$$P_1' = -\frac{6Ka^2u_0}{P_0} \sin^2 \gamma_0 \sin (\beta^* + \beta_0) \cos (\beta^* + \beta_0) \tag{60}$$

$$u_1'' + u_1 = -\frac{P_1'u_0}{P_0} - \frac{2P_1K}{P_0^3} - 2u_0(b_1' + \Lambda_{Q1}' \cos \gamma_0) + \frac{3Ka^2u_0^3}{P_0^2}[1 - 3 \sin^2 \gamma_0 \sin^2 (\beta^* + \beta_0)]. \tag{61}$$

Consider Eq. 59 first. In view of Eq. 58, it can be rewritten as

$$\alpha_1'' + \alpha_1 = -2 \sin \gamma_0 \left(\Lambda_{Q1}' + \frac{3K^2a^2}{P_0^4} \cos \gamma_0\right) \sin (\beta^* + \beta_0) - \frac{3K^2a^2}{P_0^4} \sin \gamma_0 \cos \gamma_0 [\sin \beta_0 + \sin (2\beta^* + \beta_0)]. \tag{62}$$

If this equation is solved for $\alpha_1$, it is seen that an aperiodic term proportional to $\beta^*$ must arise unless the coefficient of $\sin (\beta^* + \beta_0)$ is equal to zero. Therefore, in order to have periodicity in $\alpha$ it is necessary (and

sufficient to first order in $\mu$) to refer the motion to an orbital plane which rotates with angular rate

$$\frac{d\Lambda_0}{d\beta^*} = -\mu \frac{3K^2a^2}{P_0^4} \cos \gamma_0. \tag{63}$$

The effect described by Eq. 63 can be visualized as a slow rotation of the line of nodes in the earth's equatorial plane. For $\gamma < 90°$ the rotation is clockwise (negative) as seen from the north pole, and the effect is called the "regression of the nodes." If $\gamma > 90°$ the same treatment may be used with the understanding that the rotation is positive about the north polar axis, or it may be called a "progression of the nodes."

Relative to an orbital plane rotating in this way, $\alpha$ becomes

$$\alpha(\beta^*) = \mu \left\{ \alpha_s \sin \beta^* + \alpha_c \cos \beta^* + \frac{K^2a^2}{P_0^4} \sin \gamma_0 \cos \gamma_0 \times [\sin (2\beta^* + \beta_0) - 3 \sin \beta_0] \right\} + 0(\mu^2,\mu_3). \tag{64}$$

The choice of integration constants $\alpha_s$ and $\alpha_c$ is discussed later.

It is straightforward to show from Eqs. 58 and 60 that

$$P(\beta^*) = P_0 + \mu \left\{ P_{1k} + \frac{3K^2a^2}{2P_0^3} \sin^2 \gamma_0 \left[ \cos 2(\beta^* + \beta_0) + \frac{\epsilon}{3} \cos (3\beta^* + 2\beta_0) + \epsilon \cos (\beta^* + 2\beta_0) \right] \right\} + 0(\mu^2,\mu_3), \tag{65}$$

where $P_{1k}$ is a constant of integration.

When the results of Eqs. 64 and 65 are used in the perturbation equation for $u_1$, Eq. 61, the latter becomes

$$\begin{aligned} u_1'' + u_1 = {} & \left[ \frac{3}{8} \frac{K^2a^2}{P_0^6} \epsilon^2 \sin^2 \gamma_0 \cos 2\beta_0 + \frac{3K^2a^2}{P_0^6} \left(1 + \frac{\epsilon^2}{2}\right) \right. \\ & \times \left. \left(1 - \frac{3}{2} \sin^2 \gamma_0\right) - \frac{2KP_{1k}}{P_0^3} - \frac{2K}{P_0^2} (b_1 + \Lambda_{01}' \cos \gamma_0) \right] + 2\epsilon \\ & \times \left[ \frac{3K^2a^2}{P_0^6} \left(1 - \frac{3}{2} \sin^2 \gamma_0\right) - \frac{K}{P_0^2} (b_1 + \Lambda_{01}' \cos \gamma_0) \right] \cos \beta^* \\ & + \left[ \frac{3K^2a^2}{2P_0^6} \epsilon^2 \left(1 - \frac{3}{2} \sin^2 \gamma_0\right) \cos (2\beta^* + 2\beta_0) - \frac{3K^2a^2}{2P_0^6} \left(1 - \frac{3}{2} \epsilon^2\right) \right. \\ & \times \left. \sin^2 \gamma_0 \cos 2\beta^* \right] + \frac{5K^2a^2}{P_0^6} \epsilon \sin^2 \gamma_0 \cos (3\beta^* + 2\beta_0) \\ & + \frac{15K^2a^2}{8P_0^6} \epsilon^2 \sin^2 \gamma_0 \cos (4\beta^* + 2\beta_0). \end{aligned} \tag{66}$$

By the same argument as follows Eq. 62, a solution without secular terms can be obtained only by choosing the coefficient of $\cos \beta^*$ in Eq. 66 equal to zero. This can be done by taking

$$b_1 = \frac{3a^2K^2}{P_0^4}\left(1 + \cos^2 \gamma_0 - \frac{3}{2}\sin^2 \gamma_0\right) \tag{67}$$

which is equivalent to referring the solution to a perigee rotating in the orbital plane at a rate

$$\frac{d\beta_P}{d\beta^*} = \mu \frac{3a^2K^2}{P_0^4}\left(1 + \cos^2 \gamma_0 - \frac{3}{2}\sin^2 \gamma_0\right). \tag{68}$$

To complete the first order solution, Eq. 66 is integrated, giving

$$\begin{aligned} u_1(\beta^*) = &\left[-\frac{2P_{11}K}{P_0^3} - \frac{3K^3a^2}{P_0^6}\left(1 - \frac{\epsilon^2}{2}\right)\left(1 - \frac{3}{2}\sin^2 \gamma_0\right)\right. \\ &\left. + \frac{3K^3a^2}{8P_0^6}\epsilon^2 \sin^2 \gamma_0 \cos^2 \beta_0\right] + u_{1s} \sin \beta^* + u_{1c} \cos \beta^* \\ &- \frac{3K^3a^2}{P_0^6}\left[\frac{\epsilon^2}{6}\left(1 - \frac{3}{2}\sin^2 \gamma_0\right)\cos (2\beta^* + 2\beta_0) - \frac{1}{6}\left(1 - \frac{3\epsilon^2}{2}\right)\right. \\ &\times \sin^2 \gamma_0 \cos 2\beta^* + \frac{5}{24}\epsilon \sin^2 \gamma_0 \cos (3\beta^* + 2\beta_0) \\ &\left. + \frac{1}{24}\epsilon^2 \sin^2 \gamma_0 \cos (4\beta^* + 2\beta_0)\right]. \end{aligned} \tag{69}$$

The constants of integration $u_{1s}$ and $u_{1c}$ are discussed later.

*3.4 Initial Conditions*

At the instant the orbit is established, the satellite is described by initial values of position and velocity. It is convenient to visualize these initial conditions in terms of the following initial velocity components relative to inertial space: $V_N$, the northerly component in a plane normal to the geocentric radius at the satellite; $V_E$, the easterly component in the same plane; and $V_V$, the vertical component along the outward geocentric radius. Also, let initial position be characterized by $r_i$ and $\lambda_{Si}$. Initial longitude, though important as describing the orientation of the line of nodes relative to the sun, is omitted for the present because it does not affect the geometry of the orbit.

In these terms the initial values of the variables $u$, $P$, and $\alpha$ can be found from

$$V_N = uP(\alpha' \cos \zeta + \beta' \cos \alpha \sin \zeta) \tag{70}$$

$$V_E = uP(-\alpha' \sin \zeta + \beta' \cos \alpha \cos \zeta + \dot{\Lambda}_0 \cos \lambda_{Si}) \tag{71}$$

$$V_r = -u'P \tag{72}$$

$$r_i = 1/u \tag{73}$$

$$\sin \lambda_{si} = \sin \alpha \cos \gamma + \cos \alpha \sin \gamma \sin \beta_i. \tag{74}$$

Here $\beta_i$ is the value of $\beta$ at the initial instant $t = 0$, and all other variables are understood to be evaluated at $\beta_i$. If the power series expansions of Eqs. 51–56 are used in Eqs. 70–74, each of the latter five equations takes the form

observed position or velocity component
= a function of $(P_0, \epsilon, \gamma_0, \beta_0, \beta_i)$
+ $\mu$ times a function of $(P_1, u_r, u_e, \gamma_1, \beta_1, \alpha_r, \alpha_e)$
+ higher order terms in $\mu$. (75)

(The first order terms may also contain the parameters which appear in the zero order.)

If one equates to zero each coefficient of $\mu$ in this set of equations, five equations of the form

observed position or velocity component
= a function of $(P_0, \epsilon, \gamma_0, \beta_0, \beta_i)$ (76)

are obtained. These suffice for a determination of the five parameters listed. Also, five more equations of the form

$$0 = \text{a function of } (P_1, u_r, u_e, \gamma_1, \beta_1, \alpha_r, \alpha_e) \tag{77}$$

can be written. As usual in this treatment, no higher orders are considered. In the case of Eqs. 77, there are five equations and seven parameters to be determined, which merely means that unnecessary parameters (specifically, $\gamma_1$ and $\beta_1$) were introduced originally. However, it is convenient to keep these two parameters and to reduce the set to five by taking both $\alpha_r$ and $\alpha_e$ equal to zero, leaving $\alpha(\beta^*)$ without first harmonic terms.

From the zero approximation, one finds

$$P_0 = r_i V_H \tag{78}$$

$$\cos \gamma_0 = \frac{V_E}{V_H} \cos \lambda_{si} \tag{79}$$

$$\epsilon = \left| 1 - 2Q + \frac{V^2}{V_H^2} Q^2 \right|^{\frac{1}{2}}, \tag{80}$$

where

$$V_H = (V_E^2 + V_N^2)^{\frac{1}{2}} \tag{81}$$

$$V = (V_H^2 + V_V^2)^{\frac{1}{2}} \tag{82}$$

and where

$$Q = V_H^2 r_i / K \tag{83}$$

is a dimensionless parameter whose square root may be interpreted as the ratio of the initial horizontal velocity to the velocity that would be required to establish a circular orbit at the same value of $r_i$ if the earth were a gravitating sphere. Another parameter that arises naturally is

$$\beta_i^* = \text{arc sin } (QV_V/eV_H). \tag{84}$$

Next consider the set of equations of the form of Eq. 77 for the remaining first-order parameters $P_{1b}$, $u_{1e}$, $u_{1a}$, $\gamma_1$, and $\beta_1$. This set consists of five algebraic equations in these five unknowns:

$$\frac{P_{1b}}{P_0}\left(1 - \frac{2}{Q}\right) + \frac{u_{1e}P_0}{V_H}\cos\beta_i^* + \frac{u_{1a}P_0}{V_H}\sin\beta_i^* - \gamma_1\frac{V_N^3}{V_H^3\sin\gamma_0\cos\gamma_0} + \beta_1\frac{V_N}{V_H}\tan\lambda_{Si} = -b_1\left(\beta_i^*\frac{V_N}{V_H}\tan\lambda_{Si} + 1\right) + \frac{V_N^3}{V_E V_H}\tan\lambda_{Si}\alpha_1(\beta_i^*) - \dot{\Omega}_1\frac{V_H}{V_E}\cos\lambda_{Si} - \frac{3\sigma^3}{Q^2r_i^3}(C_1 + QC_2) \tag{85}$$

$$\frac{P_{1b}}{P_0}\left(1 - \frac{2}{Q}\right) + \frac{u_{1e}P_0}{V_H}\cos\beta_i^* + \frac{u_{1a}P_0}{V_H}\sin\beta_i^* + \gamma_1\frac{V_E^3}{V_H^3\sin\gamma_0\cos\gamma_0} - \beta_1\frac{V_E^3}{V_N V_H}\tan\lambda_{Si} = b_1\left(\beta_i^*\frac{V_E^3}{V_N V_H}\tan\lambda_{Si} - 1\right) - \frac{V_E}{V_H}\tan\lambda_{Si}\alpha_1(\beta_i^*) - \frac{3\sigma^3}{Q^2r_i^3}(C_1 + QC_2) \tag{86}$$

$$\frac{P_{1b}}{P_0} + \frac{u_{1e}P_0^3}{eK} - \frac{u_{1a}P_0^3}{eK}\cot\beta_i^* = -\frac{3\sigma^3}{Q^2r_i^3}\left(QC_3 - \frac{V_H}{V_V}C_3\right) \tag{87}$$

$$\frac{2P_{1b}}{QP_0} - \frac{u_{1e}P_0}{V_H}\cos\beta_i^* - \frac{u_{1a}P_0}{V_H}\sin\beta_i^* = \frac{3\sigma^3}{Q^2r_i^3}C_1 \tag{88}$$

$$\gamma_1\frac{V_E\sin\lambda_{Si}}{V_N\sin\gamma_0} + \beta_1 = -b_1\beta_i^* - \frac{V_E}{V_N}\alpha_1(\beta_i^*), \tag{89}$$

where

$$C_1 = \frac{e^2}{8}\sin^2\gamma_0\cos 2\beta_0 - \left(1 - \frac{e^2}{2}\right)\left(1 - \frac{3}{2}\sin^2\gamma_0\right) - \frac{e^2}{6} \times \left(1 - \frac{3}{2}\sin^2\gamma_0\right)\cos 2\beta_i^* - \frac{1}{6}\left(1 + \frac{3}{2}e^2\right)\sin^2\gamma_0\cos 2(\beta_{i*} + \beta_0) - \frac{5}{24}e\sin^2\gamma_0\cos(3\beta_i^* + 2\beta_0) - \frac{1}{24}e^2\sin^2\gamma_0\cos(4\beta_i^* + 2\beta_0) \tag{90}$$

$$C_2 = \frac{1}{2}\sin^2\gamma_0\left[\cos 2(\beta_i^* + \beta_0) + \frac{\epsilon}{3}\cos(3\beta_i^* + 2\beta_0) + \epsilon\cos(\beta_i^* + 2\beta_0)\right] \quad (91)$$

$$C_3 = \frac{\epsilon^2}{3}\left(1 - \frac{3}{2}\sin^2\gamma_0\right)\sin 2\beta_i^* + \frac{1}{3}\left(1 + \frac{3}{2}\epsilon^2\right)\sin^2\gamma_0 \times \sin 2(\beta_i^* + \beta_0) + \frac{5}{8}\epsilon\sin^2\gamma_0\sin(3\beta_i^* + 2\beta_0) + \frac{1}{6}\epsilon^2\sin^2\gamma_0\sin(4\beta_i^* + 2\beta_0). \quad (92)$$

Solving these equations, one obtains to within terms of the order of $\epsilon$,

$$\gamma_1 = -\frac{3a^2}{2Q^2r_i^2}\sin 2\gamma_0\cos^2(\beta_i^* + \beta_0) \quad (93)$$

$$\beta_1 = \frac{3a^2}{2Q^2r_i^2}\left[\cos^2\gamma_0\sin 2(\beta_i^* + \beta_0) - \left(1 - \frac{5}{4}\sin^2\gamma_0\right)4\beta_i^*\right] \quad (94)$$

$$\frac{P_{ui}}{P_0} = -\frac{3a^2}{Q^2r_i^2}(\cos^2\gamma_0 - \sin^2\lambda_{si}) \quad (95)$$

$$u_{1c}\cos\beta^* + u_{1s}\sin\beta^* = -\frac{3a^2}{Q^2r_i^2}\left\{\left[\cos^2\gamma_0 - \sin^2\lambda_{si} + \frac{1}{3}\sin^2\gamma_0\cos 2(\beta_i^* + \beta_0)\right]\cos(\beta^* - \beta_i^*) + \frac{1}{3}\sin^2\gamma_0\sin 2(\beta_i^* + \beta_0)\sin(\beta^* - \beta_i^*)\right\} \quad (96)$$

$$\beta_i = \arcsin\left(\frac{\sin\lambda_{si}}{\sin\gamma_0}\right) + \frac{3\mu}{Q^2}\frac{a^2}{r_i^2}\left[\cot^2\gamma_0\sin\lambda_{si}\sqrt{\sin^2\gamma_0 - \sin^2\lambda_{si}}\right] \quad (97)$$

$$\gamma = \gamma_0 - \frac{3\mu}{Q^2}\frac{a^2}{r_i^2}\cot\gamma_0(\sin^2\gamma_0 - \sin^2\lambda_{si}) \quad (98)$$

$$\gamma_0 = \arccos\left(\frac{V_E}{V_H}\cos\lambda_{si}\right) \quad (99)$$

$$\beta - \beta_i = \left[1 + \frac{3\mu}{Q^2}\frac{a^2}{r_i^2}\left(2 - \frac{5}{2}\sin^2\gamma_0\right)\right](\beta^* - \beta_i^*) \quad (100)$$

$$\beta_P = \beta_i - \beta_i^* + \frac{3\mu}{Q^2}\frac{a^2}{r_i^2}\left(2 - \frac{5}{2}\sin^2\gamma_0\right)(\beta - \beta_i) \tag{101}$$

$$\alpha(\beta^*) = \frac{\mu\epsilon a^2}{2Q^2 r_i^2}\sin 2\gamma_0\left[\sin(2\beta^* + \beta_0) - 3\sin\beta_0\right]. \tag{102}$$

(Although it appears from this equation that $\alpha$ has a constant component, higher order solutions would convert $\sin\beta_0$ to a term like $\sin\beta_P$, which is periodic with very low frequency.)

$$P = P_0\left\{1 - \frac{3\mu}{Q^2}\frac{a^2}{r_i^3}\left[\cos^2\gamma_0 - \sin^2\lambda_{Si} - \frac{1}{2}\sin^2\gamma_0\cos 2\beta\right]\right\} \tag{103}$$

$$\begin{aligned} u = \frac{1}{r_i Q}\Bigg\{1 &+ \epsilon\cos(\beta - \beta_P) - \frac{3\mu}{Q^2}\frac{a^2}{r_i^3}\left[\left(1 - \frac{2}{3}\sin^2\gamma_0 - \frac{5}{3}\sin^2\lambda_{Si}\right)\right. \\ &\left.\times\cos(\beta - \beta_i) + \frac{2}{3}\sin\lambda_{Si}\sqrt{\sin^2\gamma_0 - \sin^2\lambda_{Si}}\sin(\beta - \beta_i)\right] \\ &+ \frac{3\mu}{Q^2}\frac{a^2}{r_i^3}\left[\left(1 - \frac{1}{2}\sin^2\gamma_0 - 2\sin^2\lambda_{Si}\right) - \frac{1}{2}\sin^2\gamma_0\cos 2\beta\right]\Bigg\} \end{aligned} \tag{104}$$

From the form of Eq. 104, it is evident that the coefficients of the $\sin\beta$ and $\cos\beta$ terms in $u$ can be combined into an effective eccentricity which depends on both $\epsilon$ and $\mu$. However, this manipulation is not carried out until after the time behavior of $r$ and $\beta$ has been developed from $u$ and $P$.

### 3.5 *Time Behavior*

The geometry of the orbit now has been determined, described to terms within the order of $\mu^2$, $\mu\epsilon$ by Eq. 104. The time behavior can be obtained from Eq. 47, which implies $d\beta^* = u^2 P\,dt$. Letting $t = 0$ correspond to $\beta^* = \beta_i^*$ (given by Eq. 84), the time required to pass from $\beta_i^*$ to $\beta^*$ is

$$t = t_0(\beta^*) - \mu t_1(\beta^*) + 0(\mu^2), \tag{105}$$

where

$$\begin{aligned} t_0 &= \int_{\beta_i^*}^{\beta^*}\frac{dx}{u_0^2(x)P_0} \\ &= \frac{P_0^3}{K^3}\left[\left(1 + \frac{3}{2}\epsilon^2\right)(\beta^* - \beta_i^*) - 2\epsilon(\sin\beta^* - \sin\beta_i^*)\right. \\ &\qquad\left. + \frac{3}{4}\epsilon^2(\sin 2\beta^* - \sin 2\beta_i^*) + 0(\epsilon^3)\right] \end{aligned} \tag{106}$$

is obtained for the case of simple elliptic motion, and

$$t_1 = \int_{\beta_i^*}^{\beta^*} \left[ \frac{2u_1(x)}{P_0 u_0^3(x)} + \frac{P_1(x)}{P_0^2 u_0^3(x)} \right] dx$$

$$= \frac{3a^2}{P_0} \Big\{ -\frac{3}{4} \sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) + (1 - 3\sin^2 \lambda_{Si})(\beta^* - \beta_i^*)$$

$$- 2\left[ \cos^2 \gamma_0 - \sin^2 \lambda_{Si} + \frac{1}{3} \sin^2 \gamma_0 \cos 2(\beta_i^* + \beta_0) \right]$$

$$\times \sin(\beta^* - \beta_i^*) + \frac{2}{3} \sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) \cos(\beta^* - \beta_i^*)$$

$$+ \frac{1}{12} \sin^2 \gamma_0 \sin 2(\beta^* + \beta_0) \Big\} + 0(\mu\epsilon). \tag{107}$$

Combining these results,

$$\frac{K^2 t}{P_0^3} = \left[ 2\epsilon \sin \beta_i^* - \frac{3}{4} \epsilon^2 \sin 2\beta_i^* + \frac{9\mu a}{4P_0} \sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) \right]$$

$$+ \left[ 1 + \frac{3}{2} \epsilon^2 - \frac{3\mu a}{P_0} (1 - 3\sin^2 \lambda_{Si}) \right] (\beta^* - \beta_i^*)$$

$$+ 2 \left\{ -\epsilon \cos \beta_i^* + \frac{3\mu a}{P_0} \left[ \cos^2 \gamma_0 - \sin^2 \lambda_{Si} + \frac{1}{3} \sin^2 \gamma_0 \cos 2(\beta_i^* + \beta_0) \right] \right\}$$

$$\times \sin(\beta^* - \beta_i^*) - 2\left[ \epsilon \sin \beta_i^* + \frac{\mu a}{P_0} \sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) \right]$$

$$\times \cos(\beta^* - \beta_i^*) + \frac{1}{4}\left[ 3\epsilon^2 \cos 2\beta_i^* - \frac{\mu a}{P_0} \sin^2 \gamma_0 \cos 2(\beta_i^* + \beta_0) \right]$$

$$\times \sin 2(\beta^* - \beta_i^*) + \frac{1}{4}\left[ 3\epsilon^2 \sin 2\beta_i^* - \frac{\mu a}{P_0} \sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) \right]$$

$$\times \cos 2(\beta^* - \beta_i^*) + 0(\mu^2, \mu\epsilon, \epsilon^3). \tag{108}$$

The variable $\beta$ is of greater physical interest than $\beta^*$. It is related to the latter by Eq. 100 as $\beta - \beta_i = (1 + \bar{\mu})(\beta^* - \beta_i^*)$, where

$$\bar{\mu} = \frac{3\mu}{Q^2} \frac{a^2}{r_i^2} \left( 2 - \frac{5}{2} \sin^2 \gamma_0 \right). \tag{109}$$

Defining the "orbital frequency" as the non-oscillatory part of $\dot{\beta}$, it is easily shown to be equal to

$$\omega_0 = \frac{K^2}{r_i^3 V_H^3} \left[ 1 - \frac{3}{2} \epsilon^2 + \frac{3\mu}{Q^2} \frac{a^2}{r_i^2} (3\cos^2 \lambda_{Si} - \frac{5}{2} \sin^2 \gamma_0) \right]$$

$$+ 0(\mu^2, \mu\epsilon, \epsilon^3). \tag{110}$$

Then Eq. 108 gives

$$\beta - \beta_i = \omega_0 t - \left[2\epsilon \sin \beta_i^* - \frac{3}{4}\epsilon^2 \sin 2\beta_i^* + \frac{9\mu}{4}\frac{a^2}{r_i^2}\sin^2 \gamma_0 \times \sin 2(\beta_i^* + \beta_0)\right] + 2\left\{\epsilon \cos \beta_i^* - \frac{3\mu a^2}{r_i^3}\left[\cos^2 \gamma_0 - \sin \lambda_{Si} + \frac{1}{3}\sin^2 \gamma_0 \cos 2(\beta_i^* + \beta_0)\right]\right\} \sin (\beta^* - \beta_i^*) + \cdots (\text{other terms in } \beta^* - \beta_i^*). \quad (111)$$

From this, one sees that $\beta - \beta_i$ is proportional to $t$ except for small oscillatory terms. This, in turn, implies that

$$\beta^* - \beta_i^* = (1 - \bar{\mu})\omega_0 t + \text{small terms}. \quad (112)$$

The latter can be used in the right hand side of Eq. 111 to give $\beta$ purely as a function of $t$ to the order of small quantities. Actually, of course, this is but one step in an iteration process which could be carried through as many steps as desired in reversing Eq. 111 to find $\beta(t)$. The result, neglecting terms of the order of $\mu^2$, $\mu\epsilon$ and $\epsilon^3$, is

$$\beta(t) = \bar{\beta} + \omega_0 t + 2\bar{\epsilon}_1 \sin\left[(1 - \bar{\mu})\omega_0 t + \chi_1\right] - \frac{3}{4}\bar{\epsilon}_2 \sin\left[2(1 - \bar{\mu})\omega_0 t + \chi_2\right], \quad (113)$$

where

$$\bar{\beta} = \beta_i - 2\epsilon \sin \beta_i^* + \frac{3}{4}\epsilon^2 \sin 2\beta_i^* - \frac{9\mu a^2}{4r_i^3}\sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) \quad (114)$$

$$\bar{\epsilon}_1 \cos \chi_1 = \epsilon \cos \beta_i^* - \frac{3\mu a^3}{r_i^3}\left[\cos^2 \gamma_0 - \sin^2 \lambda_{Si} + \frac{1}{3}\sin^2 \gamma_0 \times \cos 2(\beta_i^* + \beta_0)\right] - 4\epsilon^2 \sin^2 \beta_i^* \quad (115)$$

$$\bar{\epsilon}_1 \sin \chi_1 = \epsilon \sin \beta_i^* + \frac{\mu a^2}{r_i^2}\sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0) + 4\epsilon^2 \sin \beta_i^* \cos \beta_i^* \quad (116)$$

$$\bar{\epsilon}_2 \cos \chi_2 = \epsilon^2 \cos 2\beta_i^* - \frac{\mu a^2}{3r_i^2}\sin^2 \gamma_0 \cos 2(\beta_i^* + \beta_0) \quad (117)$$

$$\bar{\epsilon}_2 \sin \chi_2 = \epsilon^2 \sin 2\beta_i^* - \frac{\mu a^2}{3r_i^2}\sin^2 \gamma_0 \sin 2(\beta_i^* + \beta_0). \quad (118)$$

From these results it is straightforward to return to $u = r^{-1}$ and to develop the function $r(t)$.

*3.6 Some Special Cases*

It is convenient to consider some special cases, both for their intrinsic importance in certain applications and because they furnish points of contact between the present work and that of Brouwer (2) and MacMillan (8). All of these cases are correct only to within terms of the order of $\mu^2$, $\mu\epsilon$, $\epsilon^2$.

*Equatorial Orbit*

Consider the case in which $\lambda_i = V_{Ni} = 0$. Then the value of $\gamma_0$ from Eq. 79 is zero and $\beta_i$ from Eq. 97 becomes indeterminate, as may be expected from the fact that the line of nodes from which $\beta_i$ is measured becomes indistinguishable. Without loss of generality, one may choose the reference of $\beta$ angles so that $\beta_i^* = 0$. Then the equatorial orbit is described by

$$\alpha \equiv 0 \tag{119}$$

$$P = P_0\left(1 - \frac{3\mu a^2}{Q^2 r_i^2}\right) \tag{120}$$

$$\frac{1}{r} = \frac{1}{r_i Q}\left\{1 + \frac{3\mu a^2}{Q^2 r_i^2} + \epsilon\cos\beta^* - \frac{3\mu a^2}{Q^2 r_i^2}\cos\beta^*\right\}. \tag{121}$$

This result can be checked easily against Brouwer's and MacMillan's expressions for equatorial orbits, and is found to be in agreement with them to the order terms considered.

*Polar Orbit*

Consider the case in which $V_E = 0$. To study this class of orbits, there is no loss in generality in supposing that $V_V = 0$ as well. Then $\gamma_0 = \pi/2$ and $\gamma_1 = 0$, while $\beta_i^* = \beta_1 = 0$ and $\beta_0 = \lambda_{Si}$. The plane of the orbit remains fixed in inertial space and

$$\alpha \equiv 0 \tag{122}$$

$$P = P_0\left[1 + \frac{3\mu a^2}{Q^2 r_i^2}\left(\sin^2\lambda_{Si} + \frac{1}{2}\cos 2\beta\right)\right] \tag{123}$$

$$\frac{1}{r} = \frac{1}{Qr_i}\left\{1 + \epsilon\cos\beta^* + \frac{3\mu a^2}{Q^2 r_i^2}\left[\left(\frac{1}{2} - 2\sin^2\lambda_{Si}\right)\right.\right.$$
$$+ \left(\frac{1}{4}\cos\lambda_{Si} - \frac{7}{12}\cos 3\lambda_{Si}\right)\cos\beta + \left(\frac{3}{4}\sin\lambda_{Si} - \frac{7}{12}\sin 3\lambda_{Si}\right)$$
$$\left.\left. \times \sin\beta - \frac{1}{6}\cos 2\beta\right]\right\}. \tag{124}$$

It can be seen easily that there is an algebraic mistake at the beginning of MacMillan's treatment of a polar orbit, so no comparison is possible. Brouwer treated only the case of small orbital inclinations. However, there is a brief discussion of this case by Spitzer (12) which derives $r(\beta)$ from first principles for perturbations of a circular orbit. His result is equivalent to Eq. 124 for the case $\epsilon = 0$.

**Heat Exchangers for Use in Intercontinental Missiles.**—Waste King Corporation, of Los Angeles, has announced that its heat exchangers which serve as vital components in the Atlas and Thor intercontinental missiles passed their initial Air Force qualification test. These heat exchangers—for oxygen, nitrogen or helium use—were developed to meet design specifications set by Rocketdyne, a division of North American Aviation, Inc., manufacturer of the liquid propellant rocket engines which power the two missiles.

Boyd T. Marshall, manager of Waste King Corporation's Engineering and Development Division, which developed the devices with Rocketdyne's cooperation, said the heat exchangers were specially designed for use in the Atlas and Thor engines.

Prototype quantities underwent extensive thermodynamic and functional tests on the rocket engines at Rocketdyne's Santa Susana facility.

The stainless steel Waste King heat exchangers are light weight, yet rugged enough to withstand extremely severe vibration and high internal pressures while operating at high temperatures, Marshall said.

He reported that five other Waste King heat exchangers have been designed to meet the individual requirements of rocket engines, powering virtually every intermediate and long range missile under development today. They are currently being tested by both Rocketdyne and the Sacramento Division of Aerojet General Corp.

**Effects of High Temperatures on Human Beings.**—How much heat can a human being stand before suffering a nervous breakdown? This question, critically important to engineers designing aircraft so fast that they are heated by friction with the air, was answered before a meeting of The American Society of Mechanical Engineers in San Francisco.

A technical paper written by Konrad Buettner of the University of Washington, reveals safe exposure time for a man at various temperatures. His figures indicate that under specified conditions a man covered with one centimeter of clothing might remain as long as a minute and a half in air at 900°F. without collapse. Without protective clothing he might survive a 300° temperature for the same period.

Mr. Buettner cautions that many factors which are difficult to include in experiments, such as individual health, perspiration and blood circulation must be borne in mind when considering his results. He also points out that it is difficult to secure enough reliable experimental information at extreme temperatures, since reports of individuals who survive extreme heat or cold, such as those who escape from burning buildings, usually do not provide detailed information on temperatures or other conditions.

Reporting on the effect of high temperatures on human performance Mr. Buettner describes experiments in a Link trainer, a device in which student pilots practice handling simulated aircraft controls while their actions are recorded by instruments. In experiments at temperatures between 120 and 235°, the performance of student pilots was near normal for about three quarters of the safe exposure time for a given temperature. During the last quarter of exposure time, experimenters noted a rapid increase in dizziness, faintness, mental confusion, nausea and other symptoms, followed by a sharp increase in the number of errors in handling the controls.

# ACCELERATION OF PARTICLES TO COSMIC RAY ENERGIES BY ELECTROMAGNETIC INDUCTION*

BY

W. F. G. SWANN [1]

The general theory of acceleration of charged particles by magnetic induction invokes the application of Lagrange's equations to a Lagrangian function for a charge in an external electromagnetic field defined by a vector potential $\mathbf{U}$ and a scalar potential $\varphi$. In many problems $\varphi$ is zero and the Lagrangian function becomes

$$L = -m_0c^2(1 - u^2/c^2)^{\frac{1}{2}} + \frac{e}{c}(\mathbf{U}\cdot u). \tag{1}$$

The magnetic field $\mathbf{H}$, and the electric field $E$ are given by

$$\mathbf{H} = \text{curl } \mathbf{U}; \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{U}}{\partial t}. \tag{2}$$

In many problems of changing magnetic fields it is possible, rather readily, to calculate line integrals of the electric field $\mathbf{E}$ along assigned paths, but such calculations are of no avail for calculating increase of particle energy unless we can show that the particles can describe paths such as to make use of the line integrals to the end of acquiring energy continually, at least over sufficiently long periods of time. The complexity of the particle motions is such that, usually, it is not practicable to seek solutions for energy increase by calculating the path and calculating the increase of energy as the particle traverses it. In view of the foregoing considerations it is useful to develop criteria for continual increase of energy, and theorems which give information as to lower limits of energy attained in certain specified cases. A few of these matters are discussed in the following.

## GENERAL CONSIDERATIONS PERTAINING TO THE CONTINUAL INCREASE OF ENERGY OF A PARTICLE STARTING FROM REST IN AN ELECTROMAGNETIC FIELD

It is clear that the particle, starting from rest, will move initially so as to make an acute angle with the electric field, that is, with the vector $-(1/c)(\partial \mathbf{U}/\partial t)$. As long as it continues to move so as to make an

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acute angle with the positive direction of **E**, the energy will continue to increase. It can only decrease by the motion developing a character in which the particle makes an obtuse angle with **E**, so that it has a component opposite to **E**. In order to acquire this condition, however, it would have to pass through a condition, at some point $P$, in which it moved perpendicular to **E** at the point. If there were no magnetic field, it certainly could not pass through this latter condition because, at the point $P$, the particle would have acting on it a field tending to increase the component velocity parallel to **E**, and so to bring the particle's path back to the condition in which it made an acute angle with **E**.

If there is a magnetic field when the particle is at $P$, with its path perpendicular to **E**, there will arise from this magnetic field a force $[v\mathbf{H}]/c$ perpendicular to $v$ and so parallel to **E**. This force may be in the direction of **E** or in the opposite direction, depending on the circumstances. If, however, $|\mathbf{E}| > |\mathbf{H}|$, we shall certainly have $|\mathbf{E}| > |[v\mathbf{H}]/c|$ so that even if $[v\mathbf{H}]/c$ is in the opposite direction to **E**, the resultant force will be in the direction of **E** and will bring the particle back to the condition in which its path makes an acute angle with **E**, and so the particle gains energy at $P$.

The condition $|\mathbf{E}| > |\mathbf{H}|$ as a criterion for continual gain of energy is thus a *sufficient*[2] but not always a necessary condition for continual increase of energy.

There is one exception to the above theorem. It is to be found in the case where the particle passes through a place where **E** reverses sign. In this case, the argument fails. An example is to be found in the case of a particle travelling along the general direction of propagation of a plane wave. It will be acted on by the field of the wave, which will oscillate in sign along a direction perpendicular to the general direction of the motion, and indeed, in this case the particle will alternately gain and lose energy. Of course, in the plane wave we have cited we have $|\mathbf{H}| = |\mathbf{E}|$ at all times, so that, strictly speaking, the test of our theorem is too severe. However, we are certainly on the safe side if we exclude from the theorem cases where the particle passes through a place of reversal of **E**.

### CONCERNING THE LOOPING OF A PARTICLE AROUND A LINE

If $|\mathbf{E}| > |\mathbf{H}|$, or less stringently, if the path of the particle always makes an acute angle with **E**, the particle can never describe an angle $2\pi$ around any line, $OP$, unless there is a finite line integral of **E** (at the

[2] The sufficiency of the condition for an axially symmetrical field with the $z$ and $r$ components of **U** zero, was established in the writer's first paper on this subject (*Phys. Rev.*, Vol. 43, p. 217 (1933); JOUR. FRANKLIN INST., Vol. 215, p. 273 (1933). A simplified version of the theory is given by the writer in the September issue of JOUR. FRANKLIN INST., Vol. 258, p. 205 (1954); also in a later paper, "The Acquirement of Cosmic Ray Energies by Electromagnetic Induction in Galaxies," JOUR. FRANKLIN INST., Vol. 255, p. 383 (1954).

particle) taken along the path of the particle projected in a plane perpendicular to $OP$. The reason is as follows: In the light of the hypothesis, there is a finite component of $\mathbf{E}$ in the direction of the projected path at each point thereof, and therefore, if the projected path curves through an angle $2\pi$, $\mathbf{E}$ (at the particle) projected on that path will have a finite line integral taken over the range $2\pi$.

### THE LOWER LIMIT OF THE ENERGY GAINED ALONG A PATH

If $T$ is the kinetic energy of the particle, and $W \equiv T + mc^2$, it is readily possible, by the application of Lagrange's equations, to show that

$$W_2^2 - W_1^2 = e^2(\mathbf{U}_{s2} - \mathbf{U}_{s1})^2 + \int\left[e^2\dot{s}\frac{\partial}{\partial s}(\mathbf{U}_s - \mathbf{U}_{s1})^2 - 2e^2\frac{\partial \mathbf{U}_s}{\partial t}\int \dot{s}\frac{\partial \mathbf{U}_s}{\partial s}\,dt\right]dt, \quad (3)$$

where $\mathbf{U}_s$ refers to the vector potential resolved along the direction of the path, at an arbitrary point on the path. $\dot{s}$ is the velocity along the path, subscripts (1) and (2) refer respectively to values at the point occupied at $t = 0$ and the point occupied at some later time. If we define $J$ as

$$J \equiv \dot{s}\frac{\partial}{\partial s}(\mathbf{U}_s - \mathbf{U}_{s1})^2 - 2\frac{\partial \mathbf{U}_s}{\partial t}\int \dot{s}\frac{\partial \mathbf{U}_s}{\partial s}\,dt \quad (4)$$

then, in cases where, at all instants, $J$ is positive, we can write

$$W_2^2 - W_1^2 > e^2(\mathbf{U}_{s2} - \mathbf{U}_{s1})^2.$$

In cases where the particle starts from rest and where, for large kinetic energies, $mc^2$ is neglected, this expression assumes the simple form

$$|W_2| > e|(\mathbf{U}_{s2} - \mathbf{U}_{s1})|. \quad (5)$$

This relation is of value in certain cases.

### CASE OF AN AXIALLY SYMMETRICAL FIELD, IN WHICH THE VECTOR POTENTIAL U HAS NO COMPONENT ALONG THE $r$ OR $z$ DIRECTIONS

The case cited has many interesting features, some of which have been developed by the writer in a previous paper.[3] We shall summarize a few of these.

It results from Lagrange's equations that

$$\tfrac{1}{2}\frac{dW^2}{dt} = e^2\left(\mathbf{U}_\theta - \frac{r_0\mathbf{U}_0}{r}\right)\frac{\partial \mathbf{U}_\theta}{\partial t}, \quad (6)$$

[3] "The Acquirement of Cosmic Ray Energies by Electromagnetic Induction in Galaxies," JOUR. FRANKLIN INST., Vol. 255, p. 383 (1954).

where $r_0\mathbf{U}_0$ applies at the instant and position when the particle commences to change its kinetic energy.

*Criteria for Continual Increase of Energy in the Axially Symmetrical Case*

Confining ourselves, without loss of generality, to the case where $\partial\mathbf{U}_\theta/\partial t$ is positive, we see that the *necessary* condition for continued increase of energy is

$$\mathbf{U}_\theta - \frac{r_0\mathbf{U}_0}{r} > 0 \tag{7}$$

except at $t = 0$.

A *sufficient*, but not always a necessary condition for (7) to hold is that, for all positions of the particle

$$\frac{d}{dt}(r\mathbf{U}_\theta - r_0\mathbf{U}_0) > 0$$

or, since $r_0\mathbf{U}_0$ is a constant

$$\frac{d}{dt}(r\mathbf{U}_\theta) > 0. \tag{8}$$

It can readily be shown that

$$\frac{d}{dt}(r\mathbf{U}_\theta) = \left[-\mathbf{E}_\theta - \frac{[v\mathbf{H}]_\theta}{c}\right], \tag{9}$$

where $v$ is the velocity of the particle. Since $-\mathbf{E}_\theta (= \partial\mathbf{U}_\theta/\partial t)$ is positive, a sufficient, but not always necessary, condition for (8) to be true is therefore

$$|\mathbf{E}_\theta| > |\mathbf{H}| \tag{10}$$

which is a condition already cited for the more general case in which axial symmetry is not demanded. Thus conditions (7), (8), and (10) stand in order of stringency. Condition (10) is sufficient for (7) and (8). Condition (8) is sufficient for (7), while (7) is necessary and sufficient.

*Case where, in Axial Symmetry, the Particle Starts from Rest at the Place and Instant When* $\mathbf{U}_\theta = 0$

In this case, since $\dfrac{\partial\mathbf{U}_\theta}{\partial t}$ is positive, (6) becomes

$$\frac{dW^2}{dt} = e^2\frac{\partial}{\partial t}(\mathbf{U}_\theta)^2$$

so that the energy in this case increases continually under all circumstances.

*Criterion for Absence of "Looping," in the Case of Axial Symmetry*

We have seen that (7) is the necessary and sufficient condition for continual increase of energy. Now it can readily be shown from Lagrange's equations that

$$rW\dot{\theta} = -ce(\mathbf{U}_\theta - r_0\mathbf{U}_0/r).$$

Hence, if there is continual increase of energy, $\dot{\theta}$ must always be of the same sign and finite. Thus, in such a case, the particle can never "loop," except around the $z$ axis; for if the particle should loop in any other manner, there would have to be a place where $\dot{\theta}$ was zero.

MECHANISMS IN WHICH THERE IS NO MAGNETIC FIELD IN THE SPACE SURROUNDING THE MOTIVATING CURRENTS WHEN THOSE CURRENTS ARE STEADY

Tempting problems are presented by the discussion of an infinite solenoid, and by an anchor ring winding. Here, there is no external magnetic field in the steady state, and even if the currents vary with time, the magnetic field remains small for slow variations, in spite of the existence of a very definite electromotive force over a path encircling the solenoid outside thereof, or encircling the anchor ring so as to thread it.[4] If we could neglect the external magnetic field completely on such problems, we should always have $|\mathbf{E}| > |\mathbf{H}|$ and the sufficient criterion for continual gain of energy would be assured.

Although, in cases of the type cited, there is no magnetic field in the space surrounding the motivating currents when these currents are steady, there certainly is *some* magnetic field when the currents vary with the time. This can most readily be seen from the electromagnetic equations for free space, which demand a finite value for Curl $\mathbf{H}$ and so for $\mathbf{H}$ if there is a finite value for $\partial E/\partial t$. In free space we have in fact, all of the electromagnetic vectors, $E$, $\mathbf{H}$, $\mathbf{U}$, obeying the wave equation; and, as far as our interests are concerned, we need confine our attentions only to the wave equation for $\mathbf{U}$,

$$\nabla^2\mathbf{U} - \frac{1}{c^2}\frac{\partial^2\mathbf{U}}{\partial t^2} = 0.$$

For the case of axial symmetry exemplified in Fig. 1, this equation as-

[4] In such problems, the role of the vector potential presents a more realistic picture of the origin of the electric field at a point than does the behavior of the magnetic field. Thus, in the steady state problem for a solenoid, there is no magnetic field at a point $P$ outside the solenoid, but there is a very definite vector potential. It is true that the Faraday law survives to the extent of predicting that around a path encircling the solenoid there is an electromotive force equal to the magnetic flux through the path; but this magnetic flux is confined for the most part to the area inside the solenoid. As a matter of fact, even when, as in the case of a long finite solenoid, there is a small magnetic field outside the solenoid, that field is in a direction opposite to that of the flux in the solenoid.

sumes the form

$$\frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 U_\theta}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right) = \frac{1}{c^2}\frac{\partial^2 U_\theta}{\partial t^2}. \tag{11}$$

A useful problem which serves as a basis for the discussion of other problems is that of a circular current of small size, flowing around the axis

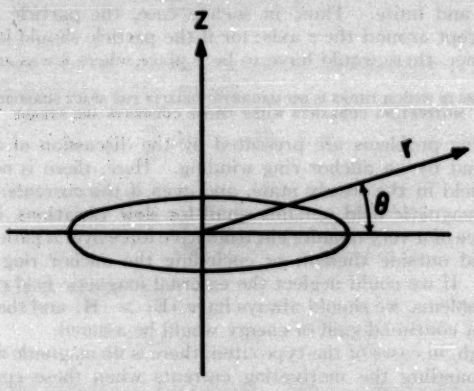


FIG. 1.

of $z$ at the origin. For the steady state case, this entity acts like a magnet of moment $\mu$ given by

$$\mu = \pi a^2 I,$$

where $a$ is the radius and $I$ the current flowing in the positive direction of $\theta$. Thus $\mu$ is in the positive direction of the axis of $z$. It can easily be verified that, for the case where $I$ varies with the time according to the law $I = I_0 f(t)$, where $f(0) = 1$, the appropriate solution of (11) is

$$U_\theta = \frac{\mu_0 r f(t - R/c)}{R^3} + \frac{\mu_0 r f'(t - R/c)}{cR^2}, \tag{12}$$

where $\mu_0 = \pi a^2 I_0$; $R^2 = z^2 + r^2$. The solution (12) can, by the combination of a number of such circular current elements, serve as the basis for the solution of a solenoid, or of a toroidal winding in which the current varies with the time.

*Case of a Toroidal Winding*

It is convenient to transpose the current ring, for which (12) is the solution, to the position shown in Fig. 2, where it now forms an element of a toroidal winding, the anchor ring of the toroid having its plane of symmetry perpendicular to the axis of $\zeta$.

Let the old axis of $z$ corresponding to (12) be in the downward direction through the paper, so that the current is in the direction shown. The plane of the paper shall be the old plane $z = 0$, and $r$ shall continue

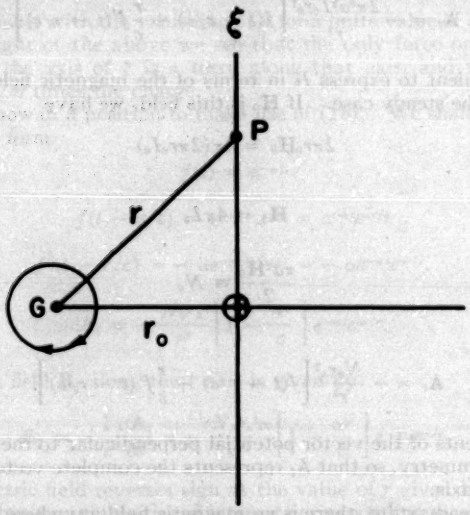


FIG. 2.

to have its original meaning. The positive direction of $\theta$ is in the direction of the arrow and the value of $\mathbf{U}_\theta$ at the point $P$ is, from (12)

$$\mathbf{U}_\theta = \frac{\mu_0}{r^2} f(t - r/c) + \frac{\mu_0}{cr} f'(t - r/c).$$

The component of $\mathbf{U}$ in the positive direction of $\zeta$ is thus

$$\mathbf{U}_\zeta = -\frac{\mu_0 r_0}{r^3} f(t - r/c) - \frac{\mu_0 r_0}{cr^2} f'(t - r/c), \tag{13}$$

where $r_0$ is as indicated.

If now we consider the toroidal winding to be made up of a large number of circular currents like that shown, each of them will make a contribution to $\mathbf{U}_\zeta$. If $I_0$ as defined above, is taken to be the current per unit length measured around the solenoid, and if $\varphi$ is measured around the $\zeta$ axis, the moment appropriate to the element of angle $d\varphi$ is $\pi a^2 I_0 r_0 d\varphi$ which replaces $\mu_0$ in (13) for the contribution of $d\varphi$ to the total vector potential $\mathbf{A}_\zeta$ along the $\zeta$ direction. Integrating with respect to $\varphi$, we find for $\mathbf{A}_\zeta$ itself the value

$$\mathbf{A}_\zeta = -\frac{2\pi^2 a^2 I_0 r_0^2}{r^3}\left[f(t - r/c) + \frac{r}{c}f'(t - r/c)\right]. \tag{14}$$

It is convenient to express $I_0$ in terms of the magnetic field inside the toroid for the steady case. If $\mathbf{H}_0$ is this field, we have

$$2\pi r_0 \mathbf{H}_0 = 4\pi(2\pi r_0 I_0)$$

so that

$$\mathbf{H}_0 = 4\pi I_0$$

writing

$$\frac{\pi a^2 \mathbf{H}_0}{2} \equiv N_0 \tag{15}$$

we have

$$\mathbf{A}_\zeta = -\frac{N_0 r_0^2}{r^3}\left[f(t - r/c) + \frac{r}{c}f'(t - r/c)\right]. \tag{16}$$

All components of the vector potential perpendicular to the $\zeta$ axis cancel, from symmetry, so that $\mathbf{A}_\zeta$ represents the complete vector potential along the $\zeta$ axis.

In the steady state, there is no magnetic field anywhere outside the toroid. Such is not the case for the non-steady state, however, for in general there is in such a case, a rate of change of vector potential and so an electric field at all points in the space around the toroid, and there is in general a rate of change of electric field which demands the existence of a magnetic field. However, along the $\zeta$ axis there is no magnetic field in a plane perpendicular to that axis; for if there were a field perpendicular to the axis at any part $P$, there would from symmetry be a magnetic flux towards or away from the axis at that point, and this is impossible. Any magnetic field in the vicinity of the $\zeta$ axis must take the form of circular lines around that axis, the situation conforming to

$$\frac{1}{c}\frac{\partial \mathbf{E}_\zeta}{\partial t} = (\text{curl } \mathbf{H})_\zeta.$$

Applying this to a small tube of radius $\sigma$ surrounding the axis we have

$$\frac{\pi\sigma^2}{c}\frac{\partial \mathbf{E}_\zeta}{\partial t} = 2\pi\sigma \mathbf{H}_\varphi,$$

where $\mathbf{H}_\varphi$ is the magnetic field in the direction of increase of $\varphi$, thus,

$$\mathbf{H}_\varphi = \left(\frac{\sigma}{2c}\right)\frac{\partial \mathbf{E}_\zeta}{\partial t}$$

and $\mathbf{H}_\varphi$ vanishes with the vanishing of $\sigma$ for a finite value of $\partial \mathbf{E}_\zeta/\partial t$.

In the light of the above we see that the only force on a charged particle on the axis of $\zeta$ is a force along that axis, and the force is $-(1/c)\partial \mathbf{A}_\zeta/\partial t$ times the charge.

We are now in a position to make use of (16). We shall choose $f(t)$ to be of the form

$$f(t) = e^{-\alpha t}$$

so that

$$f(t - r/c) = e^{-\alpha(t-r/c)} = e^{-\alpha t}e^{\alpha r/c}$$

and

$$f'(t - r/c) = -\alpha e^{-\alpha(t-r/c)} = -\alpha e^{-\alpha t}e^{\alpha r/c}.$$

Thus

$$\mathbf{A}_\zeta = -\frac{N_0 r_0^2}{r^3}\left[1 - \frac{\alpha r}{c}\right]e^{-\alpha t}e^{\alpha r/c}. \tag{17}$$

The electric field $\mathbf{E}_\zeta$ along the $\zeta$ axis is given by

$$\mathbf{E}_\zeta = -\frac{1}{c}\frac{\partial \mathbf{A}_\zeta}{\partial t} = -\frac{N_0 r_0^2 \alpha}{cr^3}\left[1 - \frac{\alpha r}{c}\right]e^{-\alpha t}e^{\alpha r/c}. \tag{18}$$

The electric field reverses sign at the value of $r$ given by $\alpha r/c = 1$. There are thus two categories of interest corresponding to $\alpha r/c < 1$ and $\alpha r/c > 1$. We shall call them cases $A$ and $B$, respectively. The second contribution (involving $\alpha r/c$) on the right-hand side of (18) is of course a close mathematical relative of the radiation field from an electric dipole, which field varies less rapidly with the distance—to the extent of one power of $r$—than does the non-radiation field, whose counterpart is the first term on the right-hand side of (18). At small distances the non-radiation term dominates, but at great distances the radiation term dominates.

### REGARDING THE ROLE PLAYED BY THE SCALE OF THE PHENOMENA

For such phenomena as occur on stars, $(1/\alpha)$ may be expected to be of the order of a few days, as in the case of sunspots, for example. For these cases $\alpha$ may be taken to be of the order $10^{-6}$ or less, so that the

quantity $c/\alpha$ is of the order $3 \times 10^{18}$ cm. This is much larger than stellar dimensions[5] so that in general for particles accelerated in stars we shall be concerned with the case where $\alpha r/c \ll 1$.

For phenomena on a galactic scale such as we encounter in the nebulae, $\alpha$ may be expected to be much smaller than $10^{-6}$. On the other hand, the dimensions available in a nebula are such that $\alpha r/c$ can approach or even exceed unity, and the scale of the space occupied by the motivating currents can afford to be correspondingly large so as to provide for significant acceleration at the great distances concerned. Thus, both cases *A* and *B* are likely to be of interest to us in cosmological speculations.

*Case A, where $\alpha r/c < 1$, so that $\mathbf{E}_\zeta$ is negative*

This corresponds to

$$(\zeta^2 + r_0^2) < c^2/\alpha^2,$$

that is, to the region

$$(c^2/\alpha^2 - r_0^2)^{\frac{1}{2}} > \zeta > -(c^2/\alpha^2 - r_0^2)^{\frac{1}{2}}$$

that is, to the region

$$\left(\frac{c^2}{\alpha^2 r_0^2} - 1\right)^{\frac{1}{2}} > \zeta/r_0 > -(c^2/\alpha^2 r_0^2 - 1)^{\frac{1}{2}}. \tag{19}$$

For $\alpha = 10^{-6}$, and $r_0 \sim 10^{11}$ for stellar dimensions

$$c^2/\alpha^2 r_0^2 = 9 \times 10^{10}.$$

Thus (19) becomes

$$\frac{c}{\alpha r_0} > \frac{\zeta}{r_0} > -\frac{c}{\alpha r_0}$$

or, for the magnitudes cited

$$3 \times 10^5 > \frac{\zeta}{r_0} > -3 \times 10^5.$$

In this region, the field is always negative, and a positive particle starting in the region moves in the negative direction and gains energy continually.

Let us consider a case where $a = 10^9$ cm.; $r_0 = 10^{10}$ cm.; $\alpha = 10^{-5}$; $\mathbf{H}_0 = 10^4$, so that, from (15), $N_0 = 1.5 \times 10^{22}$.

[5] Of course for spatial phenomena in which $(1/\alpha)$ might be of the order of 1 second, $\alpha r/c$ would be comparable with unity even for stellar dimensions.

Let us calculate the gain in energy of a proton in traveling from $-\zeta = 0$ to $-\zeta = r_0 = 10^{10}$ cm. The quantity $\alpha r/c$ will, over the whole path, always be less than $0.5 \times 10^{-8}$, so that, replacing the factor $e^{-\alpha t} e^{\alpha r/c}$ by unity, as will subsequently be justified,

$$\mathbf{E}_\zeta = -\frac{N_0 r_0^3 \alpha}{c r^3}.$$

The energy gained will be given by

$$W = -\frac{N_0 r_0^3 \alpha e}{c} \int \frac{d\zeta}{r^3}.$$

writing $-\zeta/r_0 = \tan\lambda$, we have $\pi/4$ for the upper limit of $\lambda$, and

$$W = \frac{N_0 \alpha e}{c} \int_0^{\pi/4} \frac{\sec^2 \lambda d\lambda}{(1+\tan^2 \gamma)^{\frac{1}{2}}} = \frac{N_0 \alpha e}{c} \int_0^{\pi/4} \cos\lambda d\lambda.$$

Thus

$$W = \frac{N_0 \alpha e}{c} \sin\frac{\pi}{4} = \frac{N_0 \alpha e}{1.4c} = \frac{300\, N_0 \alpha}{1.4c} \text{ e.v.}$$

$$W = 10^9 \text{e.v.}$$

If $\mathbf{E}_m$ is the field at the end of the above path, the time taken to describe the path is less than $\tau$ as given by $r_0 = \mathbf{E}_m e\tau^2/2m$, where $m$ is the relativistic mass at the end of the path. $m$ is not greatly different from the rest mass, $1.6 \times 10^{-24}$, so that it results that $r_0 = N_0 \alpha r_0^3 e\tau^2/2mr^3 c$ and, on inserting the values with $r/r_0 = 2^{\frac{1}{2}}$, we find $\tau$ of the order 1 second. Thus the replacement of $e^{-\alpha t} e^{\alpha r/c}$ by unity as above is justified.

*Case B where $\alpha r/c > 1$ over the path*

Let us write $\alpha r_0/c \equiv \eta$, and let us consider a case where $a = 10^{18}$ cm. (= 1 light year); $\mathbf{H}_0 = 10^{-3}$ gauss, (so that $N_0 = 1.5 \times 10^{22}$); $1/\alpha = 3 \times 10^{10}$, (corresponding to the decay of the motivating currents to $1/e$ of their initial values in 1000 years); $\eta = 2$, so that $r_0 = 2 \times 10^{21}$ cm. (= about 2000 light years).

The exponential factor in the expression for $\mathbf{E}_\zeta$ is composed of two factors Exp. $\alpha r/c$ and Exp. $(-\alpha t)$. We shall first examine the consequences of neglecting the second factor, so that

$$\mathbf{E}_\zeta = \frac{N_0 r_0^3 \alpha}{c r^3}\left[\frac{\alpha r}{c} - 1\right] e^{\alpha r/c}.$$

It is easy to show that $\mathbf{E}_f$ increases continually with $r$. We have

$$W > \frac{N_0 r_0^2 \alpha e}{c} \int \left[ \frac{\alpha}{cr^2} - \frac{1}{r^3} \right] d\zeta$$

over the range of integration concerned. Writing $\zeta/r_0 = \tan \lambda$

$$W > \frac{N_0 \alpha e}{c} \left[ \frac{\alpha r_0}{c} \int_0^\lambda d\lambda - \int_0^\lambda \cos \lambda d\lambda \right]$$

$$W > \frac{300\, N_0 \eta}{r_0} [\eta\lambda - \sin \lambda] \qquad \text{Electron volts.}$$

Now a proton with a velocity 99 per cent of the velocity of light has an energy of $6 \times 10^9$ e.v. It has a mass $m$ equal to 7 $m_0$. For such a proton, and with $\eta = 2$

$$2\lambda - \sin \lambda < \frac{2 \times 10^7 r_0}{2 \times N_0} < 1.5 \times 10^{-5}$$

so that $\lambda = 1.5 \times 10^{-5}$ and the corresponding value of $\zeta$ is $1.5 \times 10^{-5} r_0$. Since the minimum value of $\mathbf{E}_f$ occurs at $r = r_0$, and in this case is given by $\mathbf{E}_{min} = (N_0/r_0^2)(\alpha r_0/c)(\alpha r_0/c - 1)$ Exp. $(\alpha r_0/c)$, we have, for $\eta = 2$, $\mathbf{E}_{min} = 2(7.3) N_0/r_0^2$, and if $\tau$ is the time for the particle to reach the point $\zeta = 1.5 \times 10^{-5} r_0$, we have, with $m = 7 \times 1.6 \times 10^{-24}$

$$1.5 \times 10^{-5} r_0 > \frac{14.6\, N_0 e \tau^2}{2 r_0^2 m}.$$

Hence $\tau < 4 \times 10^5$ sec. and $\alpha\tau < 1.3 \times 10^{-5}$. Thus, the neglect of the factor Exp. $-(\alpha t)$ in the expression for $\mathbf{E}_f$ is valid for the above calculation, in which $\lambda$ is limited to the small value $1.5 \times 10^{-5}$.

We may now evaluate the total situation as follows: The least value of $\alpha r/c$ is $\alpha r_0/c$. The total time to travel a distance $\zeta$ larger than the value $1.5 \times 10^{-5} r_0$ considered above will be less than $\zeta/v + \tau$, where $v = 99c/100$, and $\tau = 4 \times 10^5$ sec. Thus, if $\zeta$ is such that $\alpha\zeta/v + \alpha\tau$ is not greater than $\alpha r_0/c$, the exponent $(\alpha r/c - \alpha t)$ in the general expression for $\mathbf{E}_f$ will always be positive. This gives

$$\zeta \leq r_0 v/c - v\tau \leq (99/100)(r_0 - 4 \times 10^5 c).$$

Since $r_0 = 2 \times 10^{21}$, the quantity $4 \times 10^5 c$ may be neglected and $\zeta$ may be permitted a value sensibly as large as $r_0$, so that the upper limit $\lambda = \pi/4$ may be used in the expression for $W$. We thus find that, in

traveling over the said range, the particle gains energy $W$ such that

$$W > \frac{300\, N_0 \eta}{r_0}\left[\frac{\pi}{4}\eta - \frac{1}{1.4}\right].$$

Putting $\eta = 2$, $N_0 = 1.5 \times 10^{21}$ and $r_0 = 2 \times 10^{21}$

$$W > 3 \times 10^{14} \text{ e.v.}$$

Of course, the numbers here cited are susceptible of enormous variations without exceeding the realm of reason, and our example is taken simply as an illustration.

**Jet Engine Exhaust Gas Temperature Measured.**—The temperature of exhaust gas from jet engines can now be measured by thermocouples which mate to a unique thermocouple harness assembly through a quick disconnect high-temperature thermocouple connector. The thermocouple harness acts as a collector ring, and can handle any number of thermocouples required on the tail cone of the engine.

This assembly is made basically from stainless steel tubing and stainless steel fittings. The tubing, which has stainless steel thermocouple outlets silver-soldered to it, houses alumel and chromel thermocouple wire spaced in ceramic insulators. When the thermocouple harness is installed on the engine and the engine is fired up, an electrical magnetic force is generated and recorded on a Millivolt meter. This meter is calibrated to register degrees of Fahrenheit directly.

In operation, the harness must withstand temperatures between 600° and 1000° F., all types of atmospheric and fuel corrosion, and engine shock and vibration. This unusual assembly is made by Airtron, Inc., Linden, N. J., from Type 430 welded stainless steel tubing supplied by the Alloy Tube Division of The Carpenter Steel Company, Union, N. J. As supplied, the tubing is 1 in. by 0.035 in. gauge with 180 grit OD polish.

Airtron, which makes a broad range of high temperature thermocouple harnesses, manufactures assemblies like this in various sizes to fit any engine tail cone configuration.

**Rapid Measurement of Metal in Solutions.**—Two University of Wisconsin chemists have developde a rapid convenient method of measuring metal in solutions one thousand times more dilute than those previously analyzed. The new electrolytic method is so precise it can measure one part of lead in five trillion parts of solution—chemically, a $10^{-9}$ moles per liter concentration. It is so sensitive that lead absorbed into the solution from glass walls of a container will upset the readings.

Developed by the UW chemistry department's Prof. Irving Shain and Research Asst. Richard D. DeMars, this analytical approach has the rather gangly title, anodic stripping voltammetry with continuously varying potential using the hanging mercury drop electrode.

As its name implies, a primary tool of the method is the unusual electrode, a tiny drop of mercury about 0.07 cm. in radius that hangs from a platinum wire. With this electrode charged negatively, the metal ions in solution slowly electroplate on the mercury drop.

"If the solution were not so dilute," explains Prof. Shain, "we could measure its concentration by a sharp, characteristic jump in current as the voltage on the electrode becomes more negative. However, in the range of concentrations we are testing—$10^{-6}$ to $10^{-9}$ moles per liter—a residual current masks these peaks."

"So instead, the UW chemists have used a process called anodic stripping," Shain said. After 15 minutes to an hour of electroplating, depending on the concentration, the voltage charge on the drop of mercury is suddenly reversed. This abrupt change forces or "strips" the accumulated metal from the mercury in just a few seconds. The concentration can then be accurately measured by a positive peak in the current-voltage curve.

# ASPECTS OF HIGH ENERGY BALLISTICS

BY

LOUIS GOLD[1]

ABSTRACT

A systematic development of the dynamics of motion appropriate in the supersonic velocity region where satellite orbits can exist, or escape from a gravitational field may arise, is presented. A more realistic treatment of the falling body problem is offered which leads to vertical flight relations that clearly indicate transition from bound to free states of motion. The results of celestial mechanics are adapted and extended toward a broader appreciation of high energy ballistics. In particular, an idealized gunnery table is constructed giving the optimal firing angle and range as well as the time of flight for firing velocities approaching the satellite region. Some elementary features of the satellite orbit are delineated.

## INTRODUCTION

The advent of long-range missiles lends considerable interest to the matter of properly treating the classic falling body problem, or more generally, ballistics at high energies. In particular, the correct formulation of gunnery is essential; fragmentary treatments are scattered in the literature. The over-all purpose of the present analysis is the systematic development of the appropriate celestial mechanics pertinent to the external ballistics of long-range missiles. It is a pity in this era of the launching of satellites one still finds antiquated presentations of this phase of physics.

Following a realistic approach to the simple subject of vertical launching, a somewhat detailed offering of the mechanics of projectiles will be presented from which a gunnery table of the optimal firing angle for attaining the greatest range possible will be deduced along with such items as the peak distance from the earth and the time of flight. Some relevant incomplete reports have appeared (1,2)[2] which essentially reflect the modest discussion contained in Cranz's book (3). The treatment to be engaged in here lends itself to an appreciation of the satellite problem and some comments will be made in this connection although several recent papers (4,5) have appeared during the progress of this work.

The discussion will be of an idealized nature. Since the chief consideration is to give a first approach to high energy ballistics, many details, some of which may be very important in practice, will be omitted. For example, the assumption of impulse launching pervades the treatment, it being supposed that the burn-out time for energizing the projectile is small compared to flight time. Furthermore, dissipa-

[1] Massachusetts Institute of Technology, Lexington, Mass.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

tive effects will be ignored—a factor which becomes of less concern as the vehicle is launched at a distance far from the earth's atmosphere where shock phenomena vanish. The spheroidal shape of the earth and its nonuniform density will tend to be manifest, but these effects will not be dealt with here although their significance for the satellite question has been pointed out (4,5). The Coriolis force is known to be of importance (6) for long-range missiles; for simplicity this too is not included. There are numerous other effects which make the field of ballistics a complex one indeed such as projectiles with spin, yaw, etc.

### MECHANICS OF VERTICAL LAUNCHING[3]

A body on the earth's surface is in a negative energy state. When sufficiently energized it can escape the potential well in which it finds itself. In the realm where the bound state can be broken it becomes necessary to formulate Newton's relation for vertical motion as

$$\frac{d}{dt}(mv) = -\frac{GMm}{x^2}. \tag{1}$$

There is no need to worry about relativistic effects for the time being, so for ready solution of this non-linear equation, let $v = v(x)$ whence

$$\frac{dv}{dx}\frac{dx}{dt} + \frac{GM}{x^2} = 0 \tag{2}$$

becomes the basic equation of motion. The solution for

$$\int_{v_0}^{v} v\,dv = -GM\int_{r_0}^{x}\frac{dx}{x^2} \tag{3}$$

incorporates the initial conditions that at $t = 0$, the velocity is $v_0$ at the earth's surface where $x = r_0$ the radius of the earth. One finds the result

$$\tfrac{1}{2}(v^2 - v_0^2) = GM\left(\frac{1}{x} - \frac{1}{r_0}\right) \tag{4}$$

which reveals at once that for $x \to \infty$ there exists a critical launching velocity $v^*$ given by

$$v^{*2} = \frac{2GM}{r_0} = 2r_0 g. \tag{5}$$

This is the so-called escape velocity.

[3] Following completion of the manuscript, it was discovered that Jeans had considered this in a problem. Our results are somewhat more detailed and in a form better suited for the purposes of the present report. (See J. H. JEANS, "Theoretical Mechanics," Boston, Ginn and Company, 1907, p. 256.)

Equation 4 readily yields the peak altitude $h = x^* - r_0$ which occurs when $v = 0$

$$\frac{1}{x^*} = \frac{1}{r_0}\left(1 - \frac{v_0^2}{v^{*2}}\right) \tag{6}$$

whence

$$h = \tfrac{1}{2}v_0^2/2g\left(1 - \frac{v_0^2}{v^{*2}}\right)^{-1}. \tag{7}$$

The low energy relation follows in the limit of $\frac{v_0}{v^*} \to 0$. On the other hand, $\frac{v}{v^*} \to 1$ delineates the escape phenomenon as $h \to \infty$.

Next the $v(t)$ relation can be calculated by utilizing (4) in (1) whereby there obtains

$$-\frac{1}{4GM}\int_0^t dt = \int_{v_0}^{v} \frac{dv}{v^4 + 2V^2v^2 + V^4}, \qquad V^2 = v^{*2} - v_0^2$$

or

$$-\frac{1}{2}\frac{t}{r_0v^{*2}} = \int_{v_0}^{v} \frac{dv}{(v^2 + V^2)^2}. \tag{8}$$

One is then confronted with three possible situations according to the value of $V^2$.

*Case 1.* $\frac{v_0}{v^*} < 1$ *or* $V^2 > 0$

$$\frac{v}{v^2 + (v^{*2} - v_0^2)} - \frac{v_0}{v^{*2}} + \frac{1}{\sqrt{v^{*2} - v_0^2}}\tan^{-1}\frac{v}{\sqrt{v^{*2} - v_0^2}} - \frac{1}{\sqrt{v^{*2} - v_0^2}}\sin^{-1}\frac{v_0}{v^*} = -\frac{1}{r_0}\left(1 - \frac{v_0^2}{v^{*2}}\right)t. \tag{9}$$

The dynamics of the vertical flight are contained in Eqs. 9 and 10; the $x(t)$ relation may be explicitly written down by using the velocity result of (4) in Eq. 9. The time of flight $\tau = 2t^*$ follows by setting $v = 0$ which produces

$$t^* = \frac{r_0/v^*}{1 - \frac{v_0^2}{v^{*2}}}\left(\frac{v_0}{v^*} + \frac{1}{\sqrt{1 - v_0^2/v^{*2}}}\sin^{-1}\frac{v_0}{v^*}\right). \tag{10}$$

The classical limit $t^* = v_0/g$ is had for $v_0/v^* \to 0$ and $t^*$ correctly goes infinite when $v_0/v^* \to 1$.

*Case 2. Critical flight* $v_0/v^* = 1$, $V^2 = 0$

The simple result follows

$$\frac{v}{v^*} = \frac{1}{1 + \frac{1}{2}\frac{t}{r_0 v^*}} \tag{11}$$

which shows the projectile floating off into space with zero velocity, that is, $t \to \infty$ means $v \to 0$.

*Case 3. Escape flights* $v_0/v^* > 1$, $V^2 < 0$

$$\frac{v}{v^2 - (v_0^2 - v^{*2})} - \frac{v_0}{v^{*2}} - \frac{1}{2\sqrt{v_0^2 - v^{*2}}} \ln \frac{\sqrt{v_0^2 - v^{*2}} + v}{\sqrt{v_0^2 - v^{*2}} - v}$$

$$+ \frac{1}{2\sqrt{v_0^2 - v^{*2}}} \ln \frac{\sqrt{v_0^2 - v^{*2}} + v_0}{\sqrt{v_0^2 - v^{*2}} - v_0} = -\frac{1}{r_0}\left(1 - \frac{v_0^2}{v^{*2}}\right) t. \tag{12}$$

In this trajectory the projectile approaches an ultimate velocity dictated by $v^2 = v_0^2 - v^{*2}$ which is the asymptotic value for $t \to \infty$.

These three cases that arise in the dynamics of vertical motion have their counterpart in the generalized problem of missile flight. Occasions will occur when reference to these results will be made. It will be helpful in treating the more involved situations to see what happens in the limit of vertical motion and/or the low energy propulsion state.

### DYNAMICS OF NON-VERTICAL LAUNCHING

The Lagrangian or Hamiltonian in polar coordinates is of course precisely that for planetary motion which is characterized by the equations of motion

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 - \frac{GMm}{r^2} \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{13}$$

One can thus take over directly the known results of celestial mechanics. For this purpose it has been found that the treatment of Pauling and Wilson (7) is particularly adapted, requiring only a slight change from electrostatic to gravitational quantities. Like most available developments of central motion, they restrict themselves to the circumstance of negative energy which corresponds to Case (1) for vertical motion.

The principal features of ballistics can be deduced from these known

relations. But it will be necessary to derive the time-dependency of the motion as it has been customary to calculate the properties of the motion from the orbits directly using Kepler's Laws. At the lower end of the energy range the orbits will not clear the earth (3) and so it is not enough to know just the periods for the elliptic orbits. The trajectories beyond the escape region require treatment of the zero and positive energy cases.

For negative energies $W < 0$, the orbits are prescribed by

$$u = \frac{1}{r} = \frac{GMm^2}{p^2} + \frac{1}{2}\sqrt{\frac{4m^2G^2m^2M^2}{p^4} + \frac{8mW}{p^2}}\sin(\theta - \theta_0) \qquad (14)$$

with the semi-major and minor axes, respectively

$$a = -\frac{GmM}{2W}, \quad b = \frac{p}{\sqrt{-2mW}} \qquad (14a)$$

and the eccentricity $\epsilon$ given by

$$1 - \epsilon^2 = -\frac{2Wp^2}{G^2M^2m^3}, \quad \epsilon < 1. \qquad (14b)$$

Now if the initial launching conditions $t = 0$, $v = v_0$, $r = r_0$, $\dot{r} = v\cos\phi$, $r\dot{\theta} = v\sin\phi$ with a firing angle $\phi$ where $\phi = 0$ for vertical flight then from

$$mr^2\dot{\theta} = p \qquad (15)$$

and

$$\frac{1}{2}m\dot{r}^2 = -\frac{p^2}{2mr^2} + \frac{GmM}{r} + W \qquad (16)$$

there obtains

$$p = mvr_0\sin\phi \qquad (17)$$

$$W = \tfrac{1}{2}m(v_0^2 - v^{*2}). \qquad (18)$$

Consequently for the ballistic problem Eqs. 14 may be re-expressed as

$$\frac{1}{r} = \frac{1}{2}\cdot\frac{1}{r_0}\cdot\frac{v^{*2}}{v_0^2}\cdot\frac{1}{\sin^2\phi}\left\{1 + \left[1 - \frac{4v_0^2}{v^{*2}}\left(1 - \frac{v_0^2}{v^{*2}}\right)\sin^2\phi\right]^{\frac{1}{2}}\sin(\theta - \theta_0)\right\}$$

$$a = \frac{\frac{1}{2}r_0}{1 - \frac{v_0^2}{v^{*2}}}, \quad b = \frac{r_0\cdot\frac{v_0}{v^*}\sin\phi}{\left(1 - \frac{v_0^2}{v^{*2}}\right)^{\frac{1}{2}}}, \quad 1 - \epsilon^2 = \frac{4v_0^2}{v^{*2}}\left(1 - \frac{v_0^2}{v^{*2}}\right)\sin^2\phi. \qquad (19)$$

The integration constant $\theta_0$ delineates the angular range $\psi$ of the missile

and is evaluable by noting that $t = 0$, $r = r_0$ at $\theta = \frac{\pi}{2}$, whence

$$\theta_0 = \cos^{-1}\left\{\frac{\frac{2v_0^2}{v^{*2}}\sin^2\phi - 1}{\left[1 - \frac{4v_0^2}{v^{*2}}\left(1 - \frac{v_0^2}{v^{*2}}\right)\sin^2\phi\right]^{\frac{1}{2}}}\right\}. \tag{20}$$

Thus since

$$\tfrac{1}{2}\psi = \pi - \theta_0$$

the missile range becomes

$$\psi = 2\sin^{-1}\left\{\frac{\frac{v_0^2}{v^{*2}}\sin 2\phi}{\left[1 - \frac{4v_0^2}{v^{*2}}\left(1 - \frac{v_0^2}{v^{*2}}\right)\sin^2\phi\right]^{\frac{1}{2}}}\right\} \tag{21}$$

which in the classical limit $v_0/v^* \to 0$ properly reduces to

$$r_0\psi = \frac{v_0^2}{g}\sin 2\phi. \tag{22}$$

The maximum range $\psi^*$ according to (22) is attained for $\phi = 45°$; clearly this is not so for $v_0/v^* \to 1$. The stationary value for $\psi$ in (21) occurs for $\phi^*$ given by

$$\sin^2\phi^* = \frac{1}{2\left(1 - \frac{v_0^2}{v^{*2}}\right)} \tag{23}$$

a result derived earlier (2) whose significance is now described. Evidently, $\phi^*$ already approaches $\pi/2$ for $v_0^2/v^{*2} \to 1/2$. It is known (3) that for $1 > v_0^2/v^{*2} \geq 1/2$ the projectile will fall into an orbit that clears the earth and hence the satellite situation. This can be seen by inserting (23) into (21) to get the optimal range

$$\psi^* = 2\sin^{-1}\frac{v_0^2/v^{*2}}{1 - v_0^2/v^{*2}} \tag{24}$$

which reveals $\psi^* \to \pi$ as $v_0^2/v^{*2} \to 1/2$. The classical result is also contained in this relation, that is, $\phi = 45°$ in (22) corresponds to $v_0/v^* \to 1$.

In the optimal trajectory defining $\psi^*$ and $\phi^*$, the uppermost position above the earth may be identified by discerning that setting $\dot{r} = 0$ in (16) specifies the point in the orbit where the vertical velocity component vanishes. This gives

$$v_0^2 - v^{*2} + v^{*2}\left(\frac{r_0}{r}\right)^* - v_0^2\sin^2\phi\left(\frac{r_0}{r}\right)^{*2} = 0$$

whence

$$\left(\frac{r_0}{r}\right)^* = \frac{v^{*2}}{2v_0^2 \sin^2\phi}\left\{1 - \left[1 - 4\left(1 - \frac{v_0^2}{v^{*2}}\right)\frac{v_0^2}{v^{*2}}\sin^2\phi\right]^{\frac{1}{2}}\right\}. \quad (25)$$

The same relation obtains from (19) when $\theta = \theta_0 - \pi/2$. The special value of $r$ becomes $r = r_0 + h$. Thus inserting (23) one finds the optimal orbit $h^* = r^* - r_0$ contained in

$$\frac{r_0}{r^*} = \frac{v^{*2}}{v_0^2}\left(1 - \frac{v_0^2}{v^{*2}}\right)\left[1 - \left(1 - \frac{2v_0^2}{v^{*2}}\right)^{\frac{1}{2}}\right]. \quad (26)$$

As $v_0^2/v^{*2} \to 1/2$, $h^*$ goes through a maximum defined by the extremum condition

$$x^2 + 2x - 1 = 0, \quad x = v_0/v^*.$$

This occurs at $v_0/v^* = 0.644$ with $(r_0/r^*)\ \min = 0.830$.

It may be further pointed out for $\phi = 0$ formula (25) reduces to the vertical launching result (6). This important finding reflects the fact that no matter what the launching angle $v_0/v^* \to 1$ means the missile will escape, that is, $(r_0/r)^* \to 0$ when

$$\left(1 - \frac{v_0^2}{v^{*2}}\right)\frac{v_0^2}{v^{*2}}\sin^2\phi = 0.$$

$\phi = 0$ doesn't apply here since in (25) there is a $\sin^2\phi$ before the brace of (25).

There still remains the stipulation of the time of flight for any firing angle as well as for the optimal range trajectory. This necessitates inquiry about the temporal properties of the flight.

### TEMPORAL FEATURES OF THE TRAJECTORIES

The time-dependency derives directly from (16) with $r = 1/u$ giving

$$t + \text{const} = -\int \frac{dU}{U^2\sqrt{2\frac{W}{m} + 2Gm\cdot U - \frac{p^2}{m^2}\cdot U^2}} \quad (27)$$

whose solution may be put into the form for $W < 0$

$$t = -\frac{r/v^*}{1 - v_0^2/v^{*2}}\left[\frac{r_0}{r} - 1 + \frac{v_0^2}{v^{*2}}\left(1 - \frac{r_0^2}{r^2}\sin^2\phi\right)\right]^{\frac{1}{2}} - \tfrac{1}{2}r_0/v^*$$

$$\times\left(1 - \frac{v_0^2}{v^{*2}}\right)^{-\frac{1}{2}}\cdot\sin^{-1}\left\{\frac{r_0/r - 2(1 - v_0^2/v^{*2})}{\frac{r_0}{r}\left[1 - \frac{4v_0^2}{v^{*2}}(1 - v_0^2/v^{*2})\sin^2\phi\right]^{\frac{1}{2}}}\right\} \text{ (Cont.)}$$

$$+\frac{r_0 v_0/v^{*2}}{1-v_0^2/v^{*2}}(1-\sin^2\phi)^{\frac{1}{2}}+\frac{1}{2}r_0/v^*(1-v_0^2/v^{*2})^{-\frac{1}{2}}$$

$$\times\sin^{-1}\left\{\frac{2v_0^2/v^{*2}-1}{[1-4v_0^2/v^{*2}(1-v_0^2/v^{*2})\sin^2\phi]^{\frac{1}{2}}}\right\}. \quad (28)$$

The time of flight obtains from this relation by employing the appropriate value of $r_0/r$ contained in Eq. 25; the explicit formula is too cumbersome to be written out.

Another approach which leads to an equivalent result that may be expressed explicitly entails establishing the period of the orbits. The solution for (27) can also be stated in the following meaningful manner:

$$\sin\left\{\frac{2(v^{*2}-v_0^2)^{\frac{1}{2}}}{v^{*2}r_0}\left[t-\frac{r}{v^{*2}-v_0^2}\right.\right.$$

$$\left.\left.\times\sqrt{(v_0^2-v^{*2})+v^{*2}\frac{r_0}{r}-v_0^2\frac{r_0^2}{r^2}\sin^2\phi}\right]\right\}$$

$$=\frac{v^{*2}\frac{r_0}{r}+2(v_0^2-v^{*2})}{\frac{r_0}{r}[v^{*4}+4(v_0^2-v^{*2})v_0^2\sin^2\phi]^{\frac{1}{2}}}+\text{const.} \quad (29)$$

Thus the left-hand member of (29) has the form $\sin(\omega\tau-\delta)$ indicating a period

$$T=\pi\frac{r_0}{v^*}(1-v_0^2/v^{*2})^{-\frac{1}{2}} \quad (30)$$

which accords with the deduction from Kepler's Laws

$$T=\frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}=\frac{\pi}{\sqrt{2}}\sqrt{\frac{r_0}{g}}(1-v_0^2/v^{*2})^{-\frac{1}{2}}. \quad (31)$$

Now invoking the law of equal areas for identical intervals of time, one has

$$\tau=T\cdot\frac{1}{2}\int_{\theta=\pi/2}^{\pi/2-\theta_0}2\cdot r^2 d\theta/\pi ab. \quad (32)$$

Thus the calculation involves solving

$$\tau=2\frac{v_0^3}{v^{*4}}\sin^3\phi$$

$$\times\int_{\pi/2}^{\pi/2-\theta_0}\frac{d\theta}{\left\{1+\left[1-4v_0^2/v^{*2}\left(1-\frac{v_0^2}{v^{*2}}\right)\sin^2\theta\right]^{\frac{1}{2}}\sin(\theta-\theta_0)\right\}^2} \quad (33)$$

with the outcome

$$\tau = \frac{2v_0^2}{v^{*2}} \frac{r_0 \sin\phi}{v_0(1 - v_0^2/v^{*2})} \left\{ \frac{v^*}{2v_0 \sin\phi} \left(1 - \frac{v_0^2}{v^{*2}}\right)^{-\frac{1}{2}} \left(\sin^{-1} \frac{\cos 2\theta_0 + b}{1 + b\cos 2\theta_0} \right.\right.$$

$$\left.\left. - \sin^{-1} \frac{\cos\theta_0 + b}{1 + b\cos\theta_0}\right) - b\left(\frac{\sin 2\theta_0}{1 + b\cos 2\theta_0} - \frac{\sin\theta_0}{1 + b\cos\theta_0}\right)\right\}, \quad (34)$$

$$b = [1 - 4v_0^2/v^{*2}(1 - v_0^2/v^{*2})\sin^2\phi]^{\frac{1}{2}}.$$

Relating $\tau$ to the angular parameter $\theta_0$ has produced a reduced expression for the time of flight. In the domain where $v_0/v^* \to 0$, $\theta_0 \to 180°$ according to (20) and so $\tau \to 2v_0/g \sin\phi$ which is the classical limit. For $\phi = 0$, one gets properly the vertical flight result (10).

In the optimal range orbit prescribed by (23) the time of flight becomes

$$\tau^* = \sqrt{2}\frac{v_0^2}{v^{*2}}\frac{r_0}{v_0}\left(1 - \frac{v_0^2}{v^{*2}}\right)^{-\frac{1}{2}} \left\{ \frac{\sqrt{2}}{2}\frac{v^*}{v_0}\left(\sin^{-1}\frac{\cos 2\theta_0 + b'}{1 + b'\cos 2\theta_0} \right.\right.$$

$$\left.\left. - \sin^{-1}\frac{\cos\theta_0 + b'}{1 + b'\cos\theta_0}\right) - b'\left(\frac{\sin 2\theta_0}{1 + b'\cos 2\theta_0} - \frac{\sin\theta_0}{1 + b'\cos\theta_0}\right)\right\}$$

$$b' = (1 - 2v_0^2/v^{*2})^{\frac{1}{2}} \quad (35)$$

where $\theta_0$ assumes the value

$$\theta_0 = \pi - \sin^{-1}\frac{v_0^2/v^{*2}}{1 - v_0^{*2}/v^{*2}}. \quad (36)$$

In the escape trajectory where $W > 0$, the solution for (27) parallels (12)

$$t + \text{const} = \frac{r}{v^*}(v_0^2/v^{*2} - 1)\left(v_0^2/v^{*2} - 1 + \frac{r_0}{r} - \frac{v_0^2}{v^{*2}}\frac{r_0^2}{r^2}\sin^2\phi\right)^{\frac{1}{2}}$$

$$- \frac{1}{2}\frac{r_0}{v^*}\left(\frac{v_0^2}{v^{*2}} - 1\right)^{-\frac{1}{2}} \ln\left\{2r(v_0^2 - v^{*2})^{\frac{1}{2}}\left[\left(v_0^2 - v^{*2} + v^{*2}\frac{r_0}{r}\right.\right.\right.$$

$$\left.\left.\left. - v_0^2\frac{r_0^2}{r^2}\sin^2\phi\right)^{\frac{1}{2}} + (v_0^2 - v^{*2})^{\frac{1}{2}}\right] + v^{*2}r_0\right\} \quad (37)$$

where the constant is prescribed by $t = 0$, $r = r_0$. This relation indicates $r \to \infty$ as $t \to \infty$. Putting $\phi = 0$ produces the vertical flight result. One can also demonstrate that for $W = 0$, Eq. 27 contains a solution which reduces to the vertical launching critical flight expressed in Eq. 11.

## CONCLUDING REMARKS

Table I summarizes the findings for the optimal range trajectory and may serve as an idealized gunnery chart. Beyond the limit of $v_0^2/v^{*2}$

$\rightarrow \frac{1}{2}$ the satellite state occurs. Precisely at $v_0^2/v^{*2} = \frac{1}{2}$, $\epsilon$ in (19) is unity and the circular orbit has the lower bound value of the satellite period which according to (30) or (31) is

$$T^* = 2\pi\sqrt{\frac{r_0}{g}}. \tag{38}$$

This bears a striking resemblance to the familiar simple pendulum formula. Employing data for the earth's radius $r_0 = 6370300$ meters and the constant $g = 980$ cm/sec$^2$, the shortest period for a satellite is about $1\frac{1}{2}$ hours. As $v_0/v^* \rightarrow 1$, the period rapidly increases until at $v_0/v^* = 1$ it becomes infinite.

TABLE I

Gunnery data for idealized flight. It emerges that the remotest objectives on the earth are within an hour flight time. The $r_0/r^*$ shows a minimum in the neighborhood of $v_0/v^* = 0.65$ corresponding to the maximum altitude of $r_0/r^* = 0.830$ at $v_0/v^* = 0.644$.

| $\frac{v_0}{v^*}$ | $\frac{v_0^2}{v^{*2}}$ | $\phi^*$ Optimal Firing Angle in Degrees | $\theta^*$ Angular Range in Degrees | $\theta_0^*$ Rotation of Elliptic Orbit in Degrees | $\frac{r_0}{r^*}$ Effective Altitude | $\tau^*$ Flight Time in Hours |
|---|---|---|---|---|---|---|
| 0.10 | 0.01 | 45.29 | 1.17 | 179.4 | 0.995 | 0.045 |
| 0.15 | 0.0225 | 45.66 | 2.64 | 178.7 | 0.989 | 0.069 |
| 0.20 | 0.04 | 46.19 | 4.78 | 177.6 | 0.980 | 0.094 |
| 0.25 | 0.0625 | 46.92 | 7.64 | 176.2 | 0.969 | 0.121 |
| 0.30 | 0.09 | 47.84 | 11.35 | 174.3 | 0.955 | 0.148 |
| 0.35 | 0.123 | 49.01 | 16.05 | 172.0 | 0.939 | 0.183 |
| 0.40 | 0.160 | 50.49 | 21.96 | 169.0 | 0.921 | 0.220 |
| 0.45 | 0.203 | 52.35 | 29.42 | 165.3 | 0.900 | 0.263 |
| 0.49 | 0.240 | 54.21 | 36.84 | 161.6 | 0.883 | 0.303 |
| 0.55 | 0.303 | 57.85 | 51.40 | 154.3 | 0.857 | 0.375 |
| 0.60 | 0.360 | 62.11 | 69.03 | 145.8 | 0.837 | 0.452 |
| 0.65 | 0.423 | 68.51 | 94.0 | 133 | 0.829 | 0.552 |
| 0.695 | 0.483 | 79.6 | 138.2 | 110.9 | 0.873 | |
| 0.70 | 0.49 | 81.95 | 147.8 | 106.1 | 0.894 | 0.682 |

Clearly to minimize the damping effects of the earth's atmosphere, vehicles should be released at some distance $d$ from the surface. The effect of the launching height may be readily incorporated into the preceding analysis as follows: The angular momentum at $t = 0$ becomes $p = m(r_0 + d)v \cdot \sin\phi$ so that the corrected escape velocity $v'^2 = \frac{r_0}{r_0 + d} v^{*2}$ and it only remains to note that $r = r_0 + h + d$ everywhere it appears in the above formulae. Thus whereas at ground levels the satellite velocity is 7900 m/sec and the escape velocity 11,050 m/sec, at firing altitudes where $d = r_0$ these values are reduced by a factor of $1/\sqrt{2}$. Elimination of atmospheric resistance occurs at heights well below $r_0$, however, so that the ground-level calculations are practicable.

I am indebted to Mrs. Billie Houghton for checking the various formulae and performing the numerical computations.

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**Roll Markers for End Face Marking.**—A new process makes it possible to mark the *end* surfaces of production parts during the machining cycle on screw machines and other multiple-station machines, thus eliminating costly secondary marking operations. Marking is done by a new standard series of automatic roll markers developed by New Method Steel Stamps, Inc., of Detroit, Mich. Designated the 900 Series by the manufacturer, the markers are easily attached to the tool mount slides of machine tools and are adaptable to automation equipment.

Radial marking of flat or angular end surfaces is the unique feature of the new process. The automatic markers are designed to permit direct in-line mounting with reference to true pitch diameter and can thus transfer numbers or characters from a conical marking die to the flat end face of the part. Conversely, it is also possible, by using a flat die, to do marking on conical or angular surfaces while holding true perspective dimensions.

Accurate marking dimensions are held since the roll markers operate on the same centers as the part being machined thus maintaining a constant alignment. Production parts already being marked with the new device include bearing races, cups and cones —various camera parts and others. Materials that can be marked include high-tensile steel, brass, aluminum, plastics, in fact any machinable material. An unusual triple safety feature is designed into the new 900 series automatic roll markers. Three separate shear pins permit the marker to fracture in stages in the event of a tooling accident and only minor replacements are necessary to restore its operation since the major components are saved.

Marking dies are standard New Method roll dies with numbers or characters cut on the true pitch diameter to suit requirements of the part or to suit marking surfaces of the part. Standard die diameters range from 0.800-inch to 4½-inches with special sizes available. Permissible marking depths range from a kiss impression to sufficient depths to prevent obliteration during later machining or grinding of the marked surface.

**Test Jig for Magnetic Cores.**—The development of a new test jig, designed for precision testing of magnetic tape wound bobbin cores, has been announced by the Electronic Instruments Division of the Burroughs Corporation, Philadelphia. The jig provides a means of applying either positive or negative current driving pulses to the core being tested and incorporates circuits permitting direct viewing on an oscilloscope of the current pulse at the point of application to the core and also the output pulse induced by the switching of the core.

The jig is available in two models, Types 8040 and 8041, the difference between models being the size of bobbin flange each will accommodate. In operation the two models are identical. The core is inserted into the jig by placing it between two pairs of sensing pins. When the cover arm is pulled down, the two pins in each pair are connected, thus forming a tight, single-turn winding on each side of the core. One for the input, the other for the output. In both models the pin spacing is adjustable in order to assure a tight loop around the core for different size bobbins.

The Types 8040 and 8041 Test Jigs have been designed as part of Burroughs Core Tester BCT 301, a complete system for individually testing tape wound cores.

# ON THE CONCEPT OF COENERGY

BY

OSMAN K. MAWARDI[1]

ABSTRACT

The concept of coenergy which appears in the study of electromechanical transducers is discussed from the point of view of thermodynamics and statistical mechanics. It is shown in this paper that either of the two approaches leads to the result that the coenergy is the Lagrangian for a system containing magnetic and/or dielectric material.

## I. INTRODUCTION

Maxwell's classical treatment (1)[2] of the generalized theory of an electrical network leans heavily on the methods of analytical dynamics. Indeed, Maxwell's starting point in the study of this problem was to point out the analogy between the kinetic and potential energy of a purely mechanical system, and the magnetic and electrostatic energy associated with the electrical network under consideration. This parallel between mechanical and electrical quantities allowed Maxwell to write at once the dynamical equations for the electrical network by means of the method of Lagrange. In these equations, the Maxwellian loop or mesh currents play the role of the generalized velocities of analytical mechanics.

A similar procedure can be extended readily to the investigation of electromechanical systems. The method (2) essentially consists in constructing a Lagrangian with mixed electrical and mechanical generalized coordinates. When no magnetic or dielectric materials are inserted in the electromagnetic field associated with the electromechanical systems, the electrical energy functions which appear in the Lagrangian are the magnetic and electrostatic energies for this electromagnetic field.

The presence of such materials does not affect the validity of the Lagrangian methods, but it requires the Lagrangian to be recast in a special form. The need for this transformation stems from the fact that the forces of electromagnetic origin which are set up in the magnetic and dielectric materials cannot be written down as equivalent to the gradient of the conventional magnetic and/or electric potential. These forces, however, are derivable from a work function related to the corresponding electromagnetic energies. The former function is often referred to in the literature as the coenergy.[3] The dependence of the

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

[3] This term was apparently introduced by C. E. Cherry, *Phil. Mag.*, Vol. 42, p. 1161 (1951).

*Note added in proof*: It was brought to the author's attention that R. A. Toupin had also introduced the Concept of Coenergy in Mechanical Systems (*J. Appl. Mech.*, Vol. 19, p. 151 (1952).

forces in question on the coenergy is readily obtained by means of an elementary demonstration which has been given in the Appendix.

The purpose of the present paper is to examine the reason for this anomaly in the form of the Lagrangian and to show that the methods of analytical mechanics really apply in general. It will be proved in the paper that the appearance of the magnetic and electric coenergy follows from a well known transformation relating the Lagrangian to the Hamiltonian of a system. It is furthermore demonstrated that the coenergy plays an important role in the theory of conversion of energy in electromechanical form. Indeed, by using thermodynamics arguments a number of state functions are uncovered and the coenergy is identified to be one of them. Another immediate consequence of this thermodynamics approach is the appearance of the conventional systems of electromechanical analogies.

The paper terminates by indicating the connection between coenergy and statistical mechanics. The reason for this part of the paper is to show that the properties of a system derived from a study of its microscopic constituents must tally with those obtained from the behavior of the system in the large, thus establishing the agreement of the results of the thermodynamics and statistical mechanics of energy conversion.

## II. MACROSCOPIC INTERPRETATION OF COENERGY

One of the fundamental axioms of the science of thermodynamics is that a thermodynamic system is completely described by its energy. This thermodynamic point of view which is reflected in the manner in which the macroscopic properties of matter in bulk are investigated will also be used here. To simplify the discussion and also to render reversible thermodynamics applicable here, the magnetic and dielectric materials included in the electromechanical systems under consideration are assumed lossless.

The investigations of this paper will be restricted to singly-excited electromechanical systems such as that of Fig. 1. They can, however,

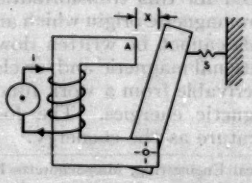


FIG. 1. Electromechanical system discussed in text.

readily be generalized to any number of degrees of freedom. A system like that of Fig. 1 having two degrees of freedom—one electrical and one mechanical—will have its operation given by two independent vari-

ables—one mechanical and one electrical—and by two dependent variables also of the same nature, that is, one mechanical and one electrical. To revert to the system of Fig. 1, the set of variables could be $(i,\Phi,f,x)$ for instance, in which $i$ stands for the current, $\Phi$ the magnetic flux in the air gap, $f$ the mechanical force, and $x$ the displacement of the armature. Another equally acceptable set is $(e,i,f,v)$ where $e$ is the voltage at the terminal of the coil, $i$ and $f$ stand for the same quantities as before, and $v$ is the mechanical velocity of the armature. If one chooses the first set and then applies the first law of thermodynamics which requires the energy to be conserved one finds that

$$dE = + id\Phi + fdx \tag{1}$$

in which $E$ is the energy in the system.

From the form of the above equation $E$ must be a function of $\Phi$ and $x$ since these two appear as the independent variables. But it is known that in a Pfaff differential (3) as occurs in Eq. 1, the replacement of one of the variables, say $\Phi$ by $i$, is obtained by the Legendre transformation

$$E_1 = E - i\Phi. \tag{2}$$

It is readily verified that this new function $E_1$ is a function of $i$ and $x$ only. Indeed, taking the derivatives on both sides of (2) it is seen that,

$$dE_1 = - \Phi di + fdx \tag{3}$$

thus establishing that $E_1 = E_1(i,x)$. Successive application of the Legendre transformation leads to two other generating functions denoted "potentials" as is customary in thermodynamics and which are functions of two variables only. Their differentials are

$$dE_2(\Phi,f) = - xdf + id\Phi \tag{4}$$

$$dE_3(i,f) = - xdf - \Phi di \tag{5}$$

while the functions themselves are

$$E_2 = E - fx = E_3 + i\Phi \tag{6}$$

$$E_3 = E - fx - i\Phi. \tag{7}$$

Whereas $E_2$ and $E_3$ have been given no special names, one identifies $E_1$ with the coenergy—except for a negative sign—when the energy of the system is *only* that in the magnetic field. The coenergy, accordingly, is the equivalent of one of the thermodynamic potentials. Actually, it is the so-called "free energy."

By inspection of the set of Eqs. 1 and 3 one finds at once

$$f = \left(\frac{\partial E}{\partial x}\right)_{\Phi}; \qquad i = \left(\frac{\partial E}{\partial \Phi}\right)_{x} \tag{8}$$

$$f = \left(\frac{\partial E_1}{\partial x}\right)_{i}; \qquad \Phi = -\left(\frac{\partial E_1}{\partial i}\right)_{x} \tag{9}$$

$$x = -\left(\frac{\partial E_2}{\partial f}\right)_{\Phi}; \qquad i = \left(\frac{\partial E_2}{\partial \Phi}\right)_{f} \tag{10}$$

$$x = -\left(\frac{\partial E_3}{\partial f}\right)_{i}; \qquad \Phi = -\left(\frac{\partial E_3}{\partial i}\right)_{f}. \tag{11}$$

Careful consideration of the relations (8) to (11) leads to the identification of four systems of electromechanical analogies and not more than four. This identification is established by observing the relative role of the variables ($i$,$\Phi$,$f$ and $x$). For instance, in (8) $f$ is the analog of $i$, which is the mobility analogy of Firestone (4). In (9), $f$ is the counterpart of $\Phi$. This is similar to the classical analogy of Maxwell. The case of (9) would have been identical to the classical case if the variables $e$, $i$, $f$ and $v$ had been chosen to described the system.

Another remark is appropriate at this point. The variables appearing in Eqs. 8 to 11 were obtained by differentiating the four potentials $E$, $E_1$, $E_2$, $E_3$. Now, it is recalled that these functions were derived from each other by a Legendre transformation. In view of the finite number of variables dealt with (*viz.*, four) it is obvious that the sequence of transformation $E \to E_1 \to E_2 \to E_3$ will form a group. In other words, the further use of a Legendre transformation, in the order just mentioned, on $E_3$ will lead one back to the function $E$. Accordingly, the system of analogies derived from them also forms a group. Several of these ideas have already been known[4] to previous investigators. This property is very strikingly demonstrated here when the results of the Eqs. 8 to 11 are tabulated (see Table I).

TABLE I.

| Independent Variables | Dependent Variables | Remark |
|---|---|---|
| $x$,$\Phi$ | $f$,$i$ | Mobility analogy |
| $x$,$i$ | $f$,$\Phi$ | Classical analogy; electrical dual of first case |
| $f$,$\Phi$ | $x$,$i$ | Mechanical and electrical dual of second case; also mechanical dual of first case; etc., |
| $f$,$i$ | $x$,$\Phi$ | Electrical dual of third case; also electrical and mechanical dual of first case, and so on. |

[4] The introduction of mixed duality has been discussed by H. Jefferson, *Wireless Engineer*, Vol. 21, p. 563 (1944) and *Phil. Mag.*, [7], Vol. 36, p. 223 (1945); and by A. Bloch, *J. Inst. Elec. Eng.* (London), Vol. 92, p. 157 (1945) and *Phil. Mag.*, [7], Vol. 36, p. 223 (1945), following the work of P. LeCorbeiller (2). The interconnection of the four systems of analogies was rec-, ognized by P. LeCorbeiller and Y. W. Young, *J. Acous. Soc. Am.*, Vol. 24, p. 643 (1952), who also pointed out the group property of the system of analogies.

A similar line of arguments can be extended to a system which is best described by the variables $e$, $q$, $f$ and $x$. Here $e$ stands for a potential difference, $q$ for a charge, while $f$ and $x$ are as before. This set of variables is specially useful when the system deals with electrostatic energies. Here also one determines four potentials $E\ (q,x)$; $E'_1(e,x)$; $E'_2(e,f)$ and $E'_3(q,f)$. Similarly, the dependent variables are given by the set

$$e = \left(\frac{\partial E}{\partial q}\right)_x; \qquad f = \left(\frac{\partial E}{\partial x}\right)_q \tag{12}$$

$$q = -\left(\frac{\partial E'_1}{\partial e}\right)_x; \qquad f = \left(\frac{\partial E'_1}{\partial x}\right)_e \tag{13}$$

$$q = -\left(\frac{\partial E'_2}{\partial e}\right)_f; \qquad x = \left(\frac{\partial E'_2}{\partial f}\right)_e \tag{14}$$

$$e = \left(\frac{\partial E'_3}{\partial q}\right)_f; \qquad x = -\left(\frac{\partial E'_3}{\partial f}\right)_q. \tag{15}$$

The above relations will later prove to be useful to the discussion. The potential $E'_1$ which appears in (13) is actually—within a negative sign—the electrical coenergy $(E - eq)$ when the energy is *only* in electrostatic form.

The aforementioned thermodynamics considerations cannot be carried any further towards the solution of the characteristics of an electromechanical system since they deal with equilibrium conditions. For the study of the dynamics of the situation one needs to resort to mechanical arguments. To that effect, the Hamiltonian of the system, $H$ is introduced. But it is known in the theory of mechanics that in a conservative system, its Hamiltonian is equal to its total energy $E$. Thus, it is possible to write for the lossless electromechanical systems discussed here

$$H = E. \tag{16}$$

The Hamiltonian, however, is a function of $2n$ variables, where $n$ is the degree of freedom of the system, the generalized momenta, $p$ and the generalized coordinates $q$ ($n$ in number). For the systems dealt with in this paper there will, therefore, be four variables $p_1$, $p_2$, $q_1$ and $q_2$. The Hamiltonian will, accordingly, display a functional relation such that

$$H = H(p_1,p_2,q_1,q_2). \tag{17}$$

Several possible selections for the four variables ($p$ and $q$) have already been discussed to some length. The dynamical characteristics for the

system, however, are given by the Lagrangian equations[5]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1,2), \tag{18}$$

where $\dot{q}_i$ are generalized velocities. These velocities are related to the momenta by expressions of the form

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{19}$$

Equations 18 imply that the Lagrangian $L$ is expected to be an explicit function of $q$, $\dot{q}$ and $t$. The theory of mechanics, however, leads to the observation that the time does *not* appear explicitly in the Lagrangian of a conservative system. Consequently, in this paper, $L$ is a function of $q$ and $\dot{q}$ only.

When dealing with a system which is purely mechanical, there is no ambiguity in identifying the generalized coordinates. Accordingly, the Lagrangian can be written down by inspection, since it turns out to be

$$L = T - V, \tag{20}$$

where $T$ is the kinetic energy of the system and $V$ the corresponding potential energy. When the system furthermore behaves linearly, in the sense that the response of the system is proportional to the excitation, it is verified that the momentum is proportional to the velocity. This constant, can, of course, be a mass or a moment of inertia. On the other hand, when the system is not linear, the momentum $p_i$ associated with $q_i$ is derived by the expression (19).

The connection between the Hamiltonian and the Lagangrian is given by a Legendre transformation which is[6]

$$L = \sum_i p_i \dot{q}_i - H. \tag{21}$$

It is readily checked that the above transformation yields a Hamiltonian which is a function of $p_i$ and $q_i$ only. This relation is always true.

The determination of the Lagrangian associated with an electromechanical system cannot always be obtained at once by inspection. The reason for this difficulty is that it is not very clear which energy of the electromagnetic field can be called kinetic and which can be labelled potential. In other words, the choice of coordinates is ambiguous. But it can be argued that the real reason for this dilemma lies in the fact

[5] The reader not familiar with analytical mechanics is referred to the standard text of E. T. Whittaker, "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," New York, Dover Publications, 4th edition, 1944.

[6] See p. 265 of Whittaker, *op. cit.* (footnote 5).

that the Lagrange equations are not invariant in form. Had they been so, the choice of the coordinates would not have mattered at all. Since the energy of a system, however, is independent of the coordinates used, it is reasonable to expect that a formulation of the equations of motion in terms of the energy will be independent of the choice of coordinates. This is actually the case and the transcription of the Lagrange equations in terms of a Hamiltonian leads to the Hamilton equations which are known[7] to be invariant in form. These equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}. \tag{22}$$

The reason the Lagrangian equations are commonly used in the electrical engineering literature instead of those of Hamilton is due to the fact that the cases of electromechanical systems considered so far have been linear. Under these conditions the difficulties discussed here are absent and no special advantage is gained from the use of Hamilton's equations. As previously mentioned, Hamilton's equations are invariable in form, that is, if the generalized coordinates and momenta $(q_i,p_i)$ are changed to a new set $(Q_i,P_i)$ through the transformation

$$\begin{aligned} Q_i &= Q_i(p_i,q_i,t) \\ P_i &= P_i(p_i,q_i,t) \end{aligned} \qquad i = 1,2 \ldots n \tag{23}$$

then one must still have

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}; \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}. \tag{24}$$

Here $K(Q_i,P_i)$ is the new Hamiltonian. Now it can be shown[8] that if a suitable function $F$ be chosen, such that $F$ is a function of the old and the new variables, as well as the time,

$$\Sigma p_i\dot{q}_i - H = \Sigma P_i\dot{Q}_i - K + \frac{d}{dt}F. \tag{25}$$

The total time derivative of $F$ can be expanded as

$$\frac{d}{dt}F = \Sigma \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial P_i} + \frac{\partial F}{\partial t}. \tag{26}$$

[7] See p. 267 ff. of Whittaker, *op. cit.*

[8] See for instance H. Goldstein, "Classical Mechanics," Cambridge, Addison-Wesley Publishing Co., 1953, p. 237 ff., in which the theory of transformation in mechanics is treated at great length.

In view of the relations (23), $F$ must be a function of $2n$ variables only (other than $t$), that is, $F$ may have the form $F(p_i,Q_i,t)$, $F(q_i,P_i,t)$, etc. . . . In particular if $F = \sum_i q_iQ_i$, one finds

$$p_i = \frac{\partial F}{\partial q_i} = Q_i$$

$$P_i = -\frac{\partial F}{\partial Q_i} = -q_i \tag{27}$$

$$H = K.$$

This case is identified with the dual transformation. The fact that the dual transformation leaves the form of Hamilton's equation unchanged is seen at once from the fact that the exchange $q_i \rightarrow p_i \rightarrow q_i$ does not alter the set

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}; \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

One is thus impressed by the arbitrariness of denoting one form of energy kinetic and another potential; even though the two roles of these two latter energies are actually reversed, the equations of motion are unaffected. The above discussion thus gives an inkling of the pitfalls that one may get into when investigating the connection between the systems of electromechanical analogy. This brings us back to the choice of analogies and the formulation of the appropriate Lagrangian.

The results obtained from thermodynamics will now be discussed in the light of the theory of analytical mechanics. Let us first consider the case of Eq. 8. This equation gave the set of relations

$$f = \frac{\partial E}{\partial X}; \qquad i = \frac{\partial E}{\partial \Phi}$$

with the understanding that $E$ the energy of the system was a function of $\Phi$ and $x$ only. If one treats $f$ and $i$ as analogs, then $x$ should be the analog of $\Phi$. Since $x$ is a mechanical coordinate, its analog $\Phi$ must also play the role of a generalized coordinate. Bearing this observation in mind, the above relations are identified as corresponding to one of the Hamilton's equation (barring the negative sign)

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}.$$

Accordingly, $f, i$ are analogs of the time derivative of a momentum. Suppose, however, one claims that the thermodynamic equations are

the counterpart of the other equation, *viz.*,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$

This would imply that $x$, $\Phi$ are now visualized as generalized momenta while $f$ and $i$ are the velocities. But one can show that either interpretation leads to the identical Lagrangian in view of the results of (25) to (27) which showed that

$$\sum p_i \dot{q}_i = -\sum Q_i \dot{P}_i. \tag{28}$$

Hence, whether $x$, $\Phi$ stand for $p_i$'s and $f$, $i$ correspond to $\dot{q}_i$'s or $f$, $i$ are $(-\dot{P}_i)$'s and $x$, $\Phi$ are $Q_i$'s one finds

$$\begin{aligned} L &= \sum p_i \dot{q}_i - H \\ &= i\Phi + fx - E. \end{aligned} \tag{29}$$

This then defines the coenergy as the electrical part of the Lagrangian obtained when the system is in a state of mechanical rest. For instance, in the transducer of Fig. 1, the magnetic coenergy is $\left( i\Phi - \int_0^{\Phi} i d\Phi \right)$. This result has already been obtained by Guggenheim (5) following similar arguments.

We now turn our attention to the remaining equations deduced from thermodynamics; these are Eqs. 9, 10 and 11. Since the functions $E_1$, $E_2$, $E_3$ are not the energy for the system, they *cannot* be Hamilton's equation. One might wonder whether this also is true of Eq. 11,

$$x = -\frac{\partial E_3}{\partial f}; \qquad \Phi = -\frac{\partial E_3}{\partial i}$$

which is seen to be the total dual of Eq. 8. The connection is readily found when one remembers that by virtue of Eq. 7

$$-E_3 = i\Phi + fx - E.$$

But this is the Lagrangian for the system. We can hence say that Eq. 11 is the Lagrange equation

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) = \dot{p}_i$$

where the usual definition of a generalized momentum (Eq. 19) has been introduced. The above equation now leads to the identification:

$x$, $\Phi$ analogs of $\dot{p}_i$
$f$, $i$ analogs of $q_i$.

One notices here that the use of the Lagrange equations does not show easily the validity of the dual transformation. On the other hand, the set of analogy is in keeping with the finding of the results of Eq. 8.

A similar interpretation for Eqs. 9 and 10

$$f = \frac{\partial E_1}{\partial x}; \quad \Phi = -\frac{\partial E_1}{\partial x} = \frac{\partial E_1}{\partial(-i)} \quad \text{and} \quad x = \frac{\partial E_2}{\partial f}; \quad i = \frac{\partial E_2}{\partial \Phi}$$

can be had, by imagining a system in which the total energy is $E_1$. This system would have a magnetic energy equal to the coenergy of the true system. This idea of introducing a hypothetical "free energy" $E_1$ is a subterfuge commonly used in thermodynamics.

If the first of the above pair of equations is considered as one of the Hamilton's equations with $H = E$, then the second pair turns out to be the Lagrange equation for the system with $L = -E_2 = fx - (i\Phi) - E_1 = fx - E$. The values found for $E_2$ are thus in accordance with that of Eq. 6. The fact that Eqs. 9 and 10 bear to each other the same connection that Eqs. 8 and 11 do, is not surprising in view of the fact that the variables of Eq. 9 are the *total* duals of those of Eq. 10.

It can also be shown that the arguments for the system of analogies apply equally well to the electrostatic case. Indeed, if Eq. 12 is identified with one of Hamilton's equations, Eq. 14 is the corresponding formulation for Lagrange's equation. The Lagrangian is then

$$L = eq + fx - E \tag{30}$$

leading to an electrostatic coenergy of $\left(eq - \int_0^q e dq\right)$. This result is not in agreement with that of Guggenheim (5). In view of the equivalence

$$e \rightarrow f; \quad q \rightarrow x$$

as obtained from Eq. 12, $e$ and $f$ are to be treated as momenta while $q$ and $x$ are to be considered as coordinates. Because of the property of the dual transformation, however, one finds that

$$\sum p_i \dot{q}_i = eq + fx$$

irrespective of the analogy $e \rightarrow f$ or $e \rightarrow x$. It is then deduced that omitting the term $eq$ from the Lagrangian of (30), as Guggenheim did, is incorrect.

One also sees that by introducing an electric "free energy" Eqs. 13 and 15 become one of Hamilton's equations and Lagrange's equation respectively.

It thus appears that the coenergy is the electrical part of the Lagrangian of an electromechanical system when magnetic and/or dielectric materials are introduced in the electromagnetic field associated with the system.

### III. STATISTICAL MECHANICS AND COENERGY

In the conventional approach to statistical mechanics an attempt is made to arrive at a prediction of the macroscopic properties of a system from a consideration of the behavior of its microscopic constituents. Accordingly, this point of view will be adopted to demonstrate the connection between the characteristics of the system such as coenergy and the molecular property of the magnetic and/or dielectric material.

One of the earliest attempts at establishing the link between the properties in the large and in the small was made by Poincaré (6) in his studies of electromagnetic transducers. The essential lines of Poincaré's arguments are briefly reproduced here for the sake of completeness.

Poincaré's approach is essentially a classical one since his investigation was completed prior to the advent of quantum mechanics. In his method the magnetic material is visualized as formed of an aggregate of Amperian loop currents. The orientation and distribution of these loops are such that the induced field due to the magnetic material is closely approximated by the resultant field of these loops. These loops are imagined to be able to move. In this way, the state of orientation of the loops will be indicative of the degree of magnetization. Although this model of a magnetic material is not in accordance with the modern atomistic theory of magnetism, it is however, adequate in exhibiting the main features of the effect of the material on the form of the Lagrangian.

Let $Q_i$ be the generalized coordinates associated with the Amperian loops and let $q_i$ be the coordinates for the external (macroscopic) parameters of the electromechanical system. If there are $n$ Amperian loops then the number of degrees of freedom for the system is $2(n + 2)$. Accordingly, the Lagrangian $L$ for the whole system is a function of $2(n + 2)$ variables *viz.*, the generalized coordinates $Q_i$ and $q_i$ as well as of their respective velocities $\dot{Q}_i$, $\dot{q}_i$. The Lagrangian instead of the Hamiltonian is used for the starting point of the discussion because the model chosen reduces the system to an aggregate of circuits in vacuum and the energy and the forces between these circuits can be expressed without ambiguity.

In writing down the Lagrangian, we consider all the energies of the components making up the system. For instance, in the transducer of Fig. 1 there are energies linked with the macroscopic parameters which represent the mechanical energies (potential and kinetic) stored in the spring and as momentum of the moving armature. In addition, there is the energy supplied by the electrical source to set up the magnetic field, which is to be differentiated from the induced field. The latter will be accounted for by the microscopic quantities. One can argue that in the present model the energy of the magnetic field (the energy

of the ether[9] as it has sometimes been called) really plays the role of a potential energy. Indeed, one can visualize it as representing the potential energy of the magnetic fluid inside which matter is imagined to be inserted.

The Lagrangian can now be written by inspection since it can be broken in two parts, one associated with macroscopic parameters and the other with microscopic parameters.

To simplify the discussion, each one of the Amperian loops will be assumed to be a circuit without resistance, electromotive force and displacement current. These loops will also be assumed to be devoid of mechanical inertia. In order to differentiate between their mechanical and electrical effects, we will denote the mechanical coordinates by $Q_{im}$ and the electrical coordinates by $Q_{ie}$. In both cases, the first letter of the subscript $i$, refers to the $i^{th}$ loop.

Since, by hypothesis, the loops have no mechanical kinetic energy, the $\dot{Q}_{im}$ will not appear in the Lagrangian. Hence it follows at once, that

$$\frac{\partial L}{\partial \dot{Q}_{im}} = 0.$$

But in view of Lagrange's equation one concludes

$$\frac{\partial L}{\partial Q_{im}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_{im}}\right) = 0.$$

In other words, all derivatives of the Lagrangian with respect to the mechanical coordinates of the loops, will vanish. Hence, it is possible to neglect for the purpose of this discussion the mechanical effects of the loops. On the other hand, since we assumed no displacement currents in any one microscopic circuit, the $Q_{ie}$'s will be absent from the Lagrangian. This leads to the result

$$0 = \frac{\partial L}{\partial Q_{ie}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_{ie}}\right).$$

Because of the removal of the mechanical variables $Q_{im}$, $\dot{Q}_{im}$, we can replace $Q_{ie}$ by $Q_i$ without causing any confusion. Consequently, we rewrite the aforementioned result as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) = 0, \quad i = 1, 2, \cdots, n \tag{31}$$

which indicates that

$$\frac{\partial L}{\partial \dot{Q}_i} = p_i = \text{constant of integration.}$$

[9] This view of ascribing one energy to the ether and another to matter when studying magnetic material has already been used by P. Debye, *Handb. Radiol.*, Vol. 6, p. 742 (1925).

As a result of the above discussion, we expect that the Lagrangian will be a function of $(n + 4)$ variables. Its form is then $L = L(q_1,q_2,\dot{q}_1,\dot{q}_2, \dot{Q}_1 \cdots \dot{Q}_n)$.

But it is well known in the theory of analytical mechanics that whenever a coordinate does not explicitely appear in the Lagrangian, it is possible to use a method, due to Routh (7), for the elimination of the generalized velocities corresponding to these absent coordinates. The latter are usually referred to as ignorable or cyclic coordinates.

The elimination of the ignorable coordinates was shown by Routh to be achieved by a transformation of the Lagrangian of the form

$$L'(q_1,q_2,\dot{q}_1,\dot{q}_2) = \sum_i p_i\dot{Q}_i - L(q_1,q_2,\dot{q}_1,\dot{q}_2,\dot{Q}_i), \tag{32}$$

where $L'$ is the new Lagrangian and $i$ the running coordinate goes from 1 to $n$, the number of Amperian loops. The form of the new Lagrangian $L'$ leaves unchanged the contribution of the coordinates $q_1$ and $q_2$.

One can also show that the new function $L'$ still satisfies the conventional Lagrange's equation. The truth of this statement is verified by noting that

$$\frac{\partial L'}{\partial Q_i} = \frac{\partial L}{\partial Q_i} = 0 \tag{33}$$

by virtue of Eqs. 31 and 32. Also, it is found that

$$\frac{\partial L'}{\partial \dot{Q}_i} = p_i - \frac{\partial L}{\partial \dot{Q}_i}. \tag{34}$$

The right-hand side of the above equation, however, vanishes as a result of the definition of $p_i$.

The new Lagrangian $L'$ also satisfies the equations of Lagrange for the macroscopic variables, that is,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = 0 \quad i = 1,2$$

which will yield the equations of motion in the *large* for the electromechanical system.

In order to arrive at the explicit evaluation of $L'$, it is required to discuss the meaning of the generalized momenta as well as that of Eq. 32. The momenta $p_i$ are associated with the electrical "kinetic" energy of the elementary loops by virtue of the definition of $p_i$. An elementary dimensional analysis, however, will indicate that the $p_i$ must be the analog of a field intensity. The proof is readily found by remembering that the Amperian loops are imagined to be idealized circuits placed in vacuum such that their aggregate produces the same induced magnetic

effects as those of the medium they replace. Hence, the contribution to the "kinetic" energy from any one loop is its magnetic energy. If $l_i$ is the effective inductance of this loop, its energy has the form $\frac{1}{2}l_i\dot{Q}_i^2$. Hence

$$p_i = \frac{\partial}{\partial \dot{Q}_i}\left(\tfrac{1}{2}l_i\dot{Q}_i^2\right) = l_i\dot{Q}_i,$$

that is, the momentum is proportional to $\dot{Q}_i$. But, the $\dot{Q}_i$ stands for the magnetic moment per unit volume. Accordingly, $p_i$ is proportional to $M$, which means dimensionally, if $l_i$ has the dimension of a susceptibility, $p_i$ has that of $H$ a magnetic field.

When the characteristics of a material are given in the form $i - \Phi$, instead of $B - H$ relation, it is an easy matter to make the necessary transformation $i \to H$, this transformation being defined completely by the geometry of the situation (that is, configuration of magnetic circuit, number of turns of coil, etc.).

We wish now to estimate the sum $\sum_i p_i\dot{Q}_i$. Since the transformation has been shown above to be one of configuration, we will assume for the moment that it is known and proceed along the lines $p_i \to H$. Actually, it will become clear in the development of this discussion that an actual knowledge of it is not required. When we consider a system such as that of Fig. 1, in which the current remains constant, then all the $p_i$'s are alike and equal to $H$. Hence

$$\sum p_i\dot{Q}_i = H\sum\dot{Q}_i.$$

It is now required to add all the elementary contribution of the $\dot{Q}_i$ in a manner more rigorous than that of the dimensional analysis. The evaluation of this summation follows a well-known method due to Langevin[10] and which is now classical. When a number of atoms are assumed to be placed in a *constant* magnetic field $H$ and when it is furthermore assumed that the interactions between these atoms may be disregarded, the Boltzmann distribution for an atom is exp $(-E_a/kT)$ where $E_a$ is the energy of an atom, $k$ Boltzmann's constant and $T$ the temperature. Assuming the intrinsic magnetic moment $\mu$ to be a constant, then we may write that part of $E_a$ which is of magnetic origin $\frac{\mu H}{kT}\cos\theta$. In the previous expression $\theta$ is the angle between the orientation of the magnetic atomic moment and the field $H$. We thus recognize the idea of orientation in the model previously mentioned and identify $\dot{Q}_i$ with $\frac{\mu H\cos\theta}{kT}$.

[10] The development of the "classical" theory of magnetism is found in several texts. See for instance L. Rosenfeld, "Theory of Electrons," Amsterdam, North-Holland Publishing Co., 1951, p. 47 ff.

To be able to find the statistical mean of $\dot{Q}_i$ and hence $\sum \dot{Q}_i$ we introduce the distribution function $\left(\exp - \frac{E_a}{kT}\right) dw_q dw_p$ where the infinitesimals are volume elements in the phase space. Now, we can write

$$E_a = E_m + \mu H \cos\theta$$

in which $E_m$ is the mechanical (translational and rotational) energy of the atom. It is remembered that in our discussion we assumed the loops had no mechanical properties. Actually, this makes no difference because the mechanical part disappears in the process of taking averages. We now define an average $M$ (or $\langle \dot{Q} \rangle$) as given by

$$M = N\mu \frac{\int \exp\left\{+\frac{\mu H \cos\theta}{kT} - \frac{Em}{kT}\right\} dw_q dw_p \cos\theta}{\int \exp\left\{+\frac{\mu H \cos\theta}{kT} - \frac{Em}{kT}\right\} dw_q dw_p}$$

$$= N\mu \left(\coth \alpha - \frac{1}{\alpha}\right) = M(H), \tag{35}$$

where $\alpha = \mu H/kT$ and $N$ is the number of atoms per unit volume. (For the numerical details in estimating the integral, the reader is referred to footnote 10.) This indicates the form of the idealized magnetization curve for the model used in this paper.

We are now in a position to evalute the new Lagrangian $L'$. First, we see that the sum $\sum p_i \dot{Q}_i = HM$ per unit volume. Also $L(q_1, q_2, \dot{q}_1, \dot{q}_2, \dot{Q}_i)$ is made up of a macroscopic mechanical energy, the energy of the ether and the energy stored in the loops. This means that

$$L = L_{\text{mech}} - \int \mu_0 H dH + H dM.$$

The first term stands for the energy stored in the spring. The second term is the energy of the ether; it appears with a negative sign because this form of energy was considered to be potential. Finally, the last term is the energy due to the induced magnetism. Substituting the above result in (32) we find that

$$L'(q_1, q_2, \dot{q}_1, \dot{q}_2) = MH - \int H dM + \int \mu_0 H dH - L_{\text{mech}}$$

$$= \int M dH + \int \mu_0 H dH - L_{\text{mech}}$$

$$= \int (M + \mu_0 H)\, dH - L_{\text{mech}}$$

$$= \int (B dH) - L_{\text{mech}}.$$

The above steps have made use of the relation between the flux density, the field and the magnetic moment per *unit volume*. When we repeat the calculation for the whole volume of the transducer together with the transformation $i$—$\Phi$ instead of $H$—$B$, we find that

$$L'(f,x,i,\Phi) = \int \Phi di - L_{\text{mech}}$$

or in terms of coenergy

$$L'(f,x,i,\Phi) = i\Phi + fx - \left(L_{\text{mech}} + \int id\Phi\right).$$

If the mechanical elements are linear then $fx - L_{\text{mech}} = L_{\text{mech}}$ for a state

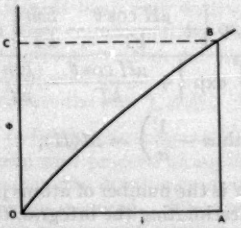


FIG. 2. Magnetization curve for the magnetic material.

of rest. By inspection of the magnetization curve of Fig. 2, we indeed identify

$$\left(i\Phi - \int id\Phi\right) = \int \Phi di$$
$$= \text{Area } OBA.$$

But this area is identified with the coenergy. The above discussion thus clearly shows the connection of the coenergy concept with the mechanics of the microscopic systems, and duplicates the results obtained by means of thermodynamic considerations which is as it should be.

A similar line of argument to that of the previous section can be used to discuss the case of the electrostatic electromechanical systems. To simplify the mathematical manipulations a highly idealized model for the dielectric material is also used. Furthermore, the geometry of the electromechanical system is chosen such that all parameters associated with the system are one dimensional. Accordingly, the dielectric is imagined to be replaced by an assembly of chains of permanent electric dipoles. These dipoles are assumed to be so constrained as to be able to be displaced only in a direction colinear with their axis. It is conceded that this model of a lossless material is far from representing the true nature of a dielectric, but it incorporates, however, some of the

salient properties which will clearly demonstrate the meaning of coenergy in the electrostatic case.

If the dipoles have no mechanical stiffness, then, they are completely described by one set of electrical coordinates $Q_i$. Any $Q_i$ stands for the dipole moment of the $i^{th}$ dipoles. Adopting the notation of the previous section, the Lagrangian of the system is

$$L = L(q_1, q_2, \dot{q}_1, \dot{q}_2, Q_1, \cdots Q_n) \tag{36}$$

in which $n$ is the number of dipoles in the dielectric slab. The number of degrees of freedom is accordingly $(n + 2)$.

By virtue of the absence of the generalized velocities $\dot{Q}_i$ the corresponding Lagrange's equations, associated with $Q_i$ coordinates, are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) = 0 = \frac{\partial L}{\partial Q_i} \qquad i = 1,2,\cdots n. \tag{37}$$

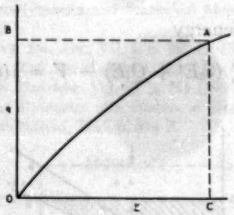


FIG. 3. Dependence of the charge $q$ on the potential difference $E$ for a slab of the dielectric.

But the dipole moments refer to microscopic parameters which are difficult to estimate and it is of interest to eliminate them from the discussion, especially that the external characteristics of the system are defined by $q_1$ and $q_2$.

The set of relations (37) actually provides the required equations by means of which the $n$ unknowns can be eliminated. Accordingly, a new Lagrangian is introduced. This is,

$$L' = \tfrac{1}{2} \sum_i (\epsilon_0 E^2 + Q_i E) + L. \tag{38}$$

In the above relation $\epsilon_0$ refers to the dielectric constant for vacuum while $E$ is the local field. Since the generalized velocities are absent from the Lagrangian, $Q_i$ will appear in the potential energy point of $L$. Since $L = T - V$, the contribution of $Q_i$ to $L$ will appear with a negative sign. This contribution can be readily evaluated in view of the specially simple model of the dielectric. Indeed, the energy of the di-

poles evaluates to $\frac{1}{2}\sum_i Q_i E$. Hence, $L'$ does not contain $Q_i$ and in essence the transformation of (38) leads to the required elimination. But it is recognized that $\frac{1}{2}\sum_i Q_i E$, the energy stored in the dielectric assumed to be inserted between two parallel plates of unit area, is graphically represented by the area $OAB$ of Fig. 3. This figure represents the dependence of the charge appearing on the plates as a function of the field strength in the dielectric assumed homogeneous. Since the intensity of the polarization vector is equal to the dipole strength per unit volume $\sum_i (\epsilon_0 E + Q_i)$ is identified with the displacement vector $D$. By an extension of Gauss's theorem $\sum_i (\epsilon_0 E + Q_i)$ is equal to $q$, the net charge appearing on the plates.

Hence, it follows that

$$L' = T + (\tfrac{1}{2}\sum_i (\epsilon_0 E^2 + Q_i E) - V)$$

and the electrostatic coenergy

$$V' = \tfrac{1}{2}\sum_i (\epsilon_0 E^2 + Q_i E) - V = \tfrac{1}{2}(qE) - V$$

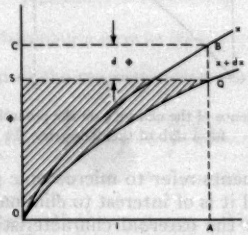


FIG. 4. Graphical construction showing change in magnetic energy for two positions $(x + \Delta x)$ of the moving armature of Fig. 1.

is graphically represented on Fig. 4 by

$$V' = \text{Area } OCAB - \text{area } OAB$$
$$= \text{Area } OCA.$$

The above discussion demonstrates the need for the use of the coenergy in describing the equations of motion of the system.

The preceding discussion has made use of classical models which are known to be artificial. A more realistic model must be along quantum-mechanical lines. Obviously, the latter approach is more correct. But it is doubtful that it would affect the validity of the general results which

have been obtained here. It can also be argued that the classical mechanics considerations, if not rigorously proving the connection between the coenergy and the properties of the material, have at least established the plausibility of such a connection.

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## APPENDIX

### *Graphical Proof for the Coenergy*

The mechanical force due to the magnetic field in the air gap $AA$ (Fig. 1) is balanced by the tension in the spring $S$. The magnitude of this force can be evaluated by means of the principle of virtual work. Consideration of the energy balance for an elementary virtual displacement of this system indicates that the incremental energy input from the source must be equated to the elementary mechanical work done by the spring and to change in the stored magnetic energy.

In terms of the flux $\Phi$, of the mechanical force $f$, of the input current $i$ and the source voltage $e$, the above condition is formally written as,

$$-eidt = fdx + d\left(\int_0^\Phi id\Phi\right).$$

The left-hand side of the above equation contains the time $dt$, which is the time during which the virtual work takes place. Introducing the relation between $e$ and $\Phi$ and assuming the virtual work to be done at constant current the above relation transforms to

$$id\Phi = fdx + d\int_0^\Phi id\Phi$$

and

$$fdx = id\Phi - d\left(\int_0^{\Phi} id\Phi\right)$$

$$= d\left(\int_0^{i} \Phi di\right) = d \text{ (Coenergy)}.$$

Since $id\Phi$ is represented by the area $BCSQ$ (Fig. 4), inspection of Fig. 4 readily indicates that the change $OBQ$ is identical with $d\left(\int_0^{i} \Phi di\right)$ that is, the force is derivable from the rate of change of the coenergy.

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# TELEPHONE EXHIBIT

BY
THOMAS COULSON

Ever since man experienced the need to expand his communications beyond the range of his primitive community, he applied his mind to the problems of sending messages over distances. Until the beginning of the 19th Century messages were transmitted by a variety of means, by signal fires, drums, runners, post-riders, and pigeons. Several attempts were made to establish a system of visual telegraphs, but it was not until the year 1794 that Claude Chappe developed a practical semaphore that answered this purpose. It was claimed that a message could be transmitted, weather permitting, from Paris to Toulon, a distance of 475 miles, in twelve minutes by this method.

But it is only during little more than a century that science has been able to provide a means to communicate instantly words, ideas, and pictures. The impetus came from the development of the electric telegraph. Following the rapid expansion of the telegraph came demands for a simple system of communication which would enable a person to communicate directly with another. The expansion of business and the growth of cities stimulated the need and presented a challenge to inventors. Eventually, Alexander Graham Bell developed a practical telephone.

Bell's greatest achievement was not so much in the making of an instrument as the discovery and understanding of the principles underlying the use of electricity to transmit speech. If the telephone had to fulfill the destiny contemplated for it as a means of personal communication, methods of intercommunication had to be developed so that any one telephone might be connected with another. A gigantic system had to be built around the three basic factors: 1. the telephone receiver and transmitter, 2. the connecting lines and 3. the central switching point. As the system grew to provide long distance service it became necessary to establish connecting facilities between central offices, such as toll lines, cables, and radio relays.

Within a few years the system had expanded to such an extent that city telephone poles carried such festoons of wires it looked as if the saturation point had been reached. This situation was relieved by the development of cables carrying bundles of wire which were buried under the streets. The recently developed coaxial cable permits several hundred telephone conversations to pass over it simultaneously.

A still more recent development dispenses with cables. Electric waves of superhigh frequency are beamed to and relayed through the air by installations in towers situated some 25 miles apart. By this

means telephone conversations and television programs are transmitted across country. Few people realize they see and hear their television network programs by virtue of the telephone system.

The continuous research in science and engineering pursued by the telephone companies has rendered possible a wide range of improvements and refinements in the facilities to make the service faster, more efficient and reliable, and labor-saving. The Bell Telephone Company's exhibit in the Museum has been entirely remodelled to furnish visitors with an instructive and entertaining picture of the most recent developments in the telephone field. Its insight into the "behind-the-scenes" operations of the most popular system of communication will be welcomed. Those former visitors who console themselves with the reflection that they have seen the exhibit will find that its new features will justify a return. Those who contemplate attending in a group will, naturally enough, use the telephone to make a reservation. The number to call is Locust 4-3600, and ask for Extension 201.

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# THE ADVANTAGES OF FLOW CONTROL VALVES, OVER ORIFICES AND CAPILLARIES, IN COMPENSATING HYDROSTATIC BEARINGS

BY

ALFRED M. LOEB [1] AND HARRY C. RIPPEL [1]

A hydrostatic bearing is a bearing in which the pressurized lubricant required to separate the moving parts is supplied from an external source. Assuming that the lubricant is of constant viscosity, these bearings obey the following laws:

Law 1 The flow of lubricant to the bearings is proportional to the load and to the cube of the film thickness.

Law 2 The load supported by a hydrostatic bearing is proportional to the pressure in the recess.

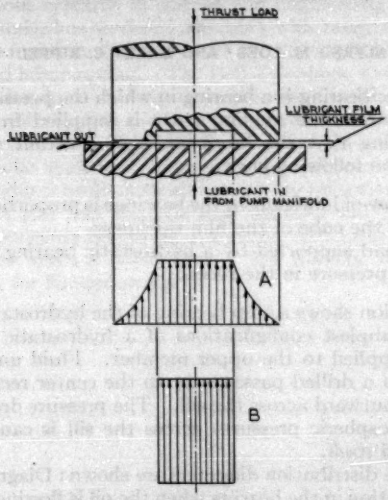
The illustration shows a step bearing of the hydrostatic type. This is one of the simplest configurations of a hydrostatic bearing. The thrust load is applied to the upper member. Fluid under pressure is pumped through a drilled passageway to the center recess and is then allowed to flow outward across the sill. The pressure drop (from recess pressure to atmospheric pressure) across the sill is caused by viscous shear in the fluid itself.

Two pressure distribution diagrams are shown: Diagram A indicates pressure distribution in the bearing when the oil is flowing across the sill, and the applied load is being supported on a film of oil. The supporting force is generated by the supply pressure exerted on the area above the recess and, also, by a decaying pressure distribution across the sill. Since the bearing surfaces are in metal-to-metal contact when the pump pressure is first applied, it is exerted only on the area immediately above the recess. This area being small, the pressure must be higher for initial lift (diagram B) than that required to maintain the bearing in its lifted state. Although the initial lift pressure is theoretically limited to the high pressure recess, local deformities and irregularities in the bearing surface reduce this pressure somewhat from the theoretical value.

When a single hydrostatic bearing is supplied from a constant displacement pump, all of the flow from the pump is delivered to the high pressure recess. In this instance, the pump pressure will automatically rise to the pressure required for initial lift. However, should two or more bearings be supplied from the same pump, because of minute differences among the bearings, one is likely to lift before the others. If this occurs, the manifold pressure decreases and will be determined by the load on the lifted bearing, according to Law 2. The entire out-

[1] Research Engineer, Friction and Lubrication Section.

put of the pump will then be discharged through the lifted bearing, and the other bearings will never lift. (The film thickness in the lifted bearing will adjust to the required amount according to Law 1.)




To limit the flow to the first lifted bearing, various methods of compensation have been employed to allow the pump pressure to rise to the value necessary to lift the other bearings in the circuit.

The earliest and perhaps one of the simplest methods was to put an orifice or capillary (hydraulic resistance) in series with each of the bearings. Thus, when the first bearing lifts, and flow through that bearing begins, the pressure in the manifold drops. As the flow to the bearing increases, the pressure drop across the capillary increases. It follows that the manifold pressure increases since the pressure in the recess remains fixed at the value necessary to support the load on the bearing.

The problem of initial lift is further complicated when the loads and geometry of the bearings are different. It is even further compounded when the load can be zero or negligible on one of the bearings. The greater the difference in load and the greater the number of bearings connected to a given pump, the greater must be the hydraulic resistance in series with each bearing. When all bearings are lifted, this greater resistance consumes considerable power.

When flow control valves are used as compensating elements for several bearings supplied from the same pump, the flow to the first

lifted bearing is automatically limited to, and maintained at, a pre-set level. Therefore the remainder of the pump output *must* go to the other bearings, which are also valve-compensated. It can easily be seen that all bearings will automatically lift if the pump flow is held constant and exceeds that required by the bearings.

In a valve-compensated bearing circuit, the pump pressure need be only 100 psi. higher than the maximum pressure required in the recess for satisfactory operation of the bearings. This results in a considerable saving of power over orifice or capillary compensation, where the pressure drop needed for proper compensation is often as high as 1000 psi.

The stiffness of a hydrostatic bearing is increased by using flow control valves to compensate the system. In an orifice-compensated bearing when the load increases, a decrease in flow must occur along with the change in film thickness; lesser flow will effect the necessary decrease in pressure drop across the compensating orifice and an increase in recess pressure. The same is true of capillary compensation. In other words, to get a pressure rise in the recess of a capillary or orifice-compensated bearing, there must be associated with this pressure rise a decrease in flow. However, if flow control valves are used, the flow to each bearing or bearing recess is held constant by the valve and is independent of recess pressure. For a given decrease in film thickness, if the flow is held constant rather than being allowed to decrease, a greater load increase can be supported. Thus the flow-control-valve-compensated bearing is stiffer than a similar orifice- or capillary-compensated bearing.

An additional advantage—and a highly desirable one—is that the characteristics of valve-compensated bearings (flow, load carrying capacity and film thickness) are more easily calculated. The ability to remove the flow term as a common, constant factor from the complicated equations which express the properties of these bearings simplifies the analytical work considerably.

Although this discussion has been limited to bearings of the single recess type, it is equally applicable to multiple-recess bearings. In the latter case, there is interaction among the bearing recesses, and control valves are even more advantageous.

Flow control valves that keep the flow of fluid constant regardless of inlet or outlet pressure are available from several manufacturers of hydraulic components. The staff of the Friction and Lubrication Section of the Laboratories have been successful in applying such valves to both single and multiple-recess hydrostatic bearings. A full-length technical discussion of this work is now being prepared by the authors.

THE BIOCHEMICAL RESEARCH FOUNDATION

ELLICE McDONALD, FOUNDER
WILLIAM G. BATT, DIRECTOR

***Abstract of* Diet-Age Pattern for Hepatic Enzyme Activity.**—MORRIS H. ROSS AND WILLIAM G. BATT (*The Journal of Nutrition*, **61**, 39–49 (1957)). The relationship has been investigated between activity of hepatic enzymes and age of the rat and between hepatic enzymes and diet. The activity levels of alkaline phosphatase, histidase, D-amino acid oxidase and cathepsin have been adjusted from that of one age to that of another age by means of diet. Diets containing large proportions of casein altered the levels of activity from those of young rats to those of older rats while diets containing small proportions of casein altered the levels of activity of older rats to those of more youthful rats. At each age, these hepatic enzymes responded to diet to the same degree. It is suggested that a diet-age pattern exists for all hepatic enzyme activity.

## BOOK REVIEWS

BORON, CALCIUM, COLUMBIUM AND ZIRCONIUM IN IRON AND STEEL, by R. A. Grange, F. J. Shortsleeve and D. C. Hilty, W. O. Binder, G. T. Motock and C. M. Offenhauer. 533 pages, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1957. Price, $14.00.

This text is the fourth in the Alloys of Iron New Monograph Series. It is the first book of the series to cover more than one alloying element. The coverage on each of the elements is adequate, and the reader should find it more convenient to be able to refer to one text instead of four.

The full effect of boron in steel can be realized with amounts as low as 0.0005 per cent. Thus, the consumption of this element in steel is not apt to cause a shortage of boron. However, small amounts of boron affect the heat treating and engineering properties of steel remarkably. The first part of the book covers these effects in detail.

The second part deals with calcium. Since calcium is almost completely insoluble in liquid or solid steel, it is not used as an alloying element. Its importance in the steel industry results in its control of oxygen, sulfur, and inclusions. Its importance in the iron industry lies primarily in its effect on microstructure and properties of gray cast iron. Forty pages are devoted to those subjects. There is a very complete list of references.

The third part of the text concerns columbium. There is considerable controversy in the engineering and scientific field concerning the name of this element. The editor points out that the name "columbium" was used in this book in preference to "niobium" as the result of a poll of a large number of prominent metallurgists in the United States. The name, "niobium," was approved by the International Union of Pure and Applied Science and the American Chemical Society. It is unfortunate that an agreement cannot be reached. Perhaps, it will be necessary to compromise on a third term such as "colobium" or "niumbium." In any event forty per cent of the book is devoted to the various aspects of columbium in iron and steel. This section is divided into eight different chapters. In the copy reviewed by the writer, the first eight pages of chapter 2, part 3 were collated upside down.

About twenty per cent of the book deals with zirconium in iron and steel. The subject is well covered and is presented lucidly. There are several books recently published on zirconium. In view of the comprehensive coverage by these books, it seems that the eleven pages in this text devoted to such subjects as history, sources, extraction, production, etc. are of little value.

There are both a subject and a name index. The references are listed at the end of each section and are grouped into years.

This book is well written and edited. It is a very worthwhile addition to the metallurgist's library.

R. L. SMITH
*The Franklin Institute Laboratories*

INSTRUMENTAL ANALYSIS, by Paul Delahay. 384 pages, diagrams, 6 × 9¼ in. New York, The Macmillan Co., 1957. Price, $7.90.

Scientific instruments in the modern world serve to supplement the human senses and powers of observation in areas in which the senses alone are inadequate. That it is important for the chemist to know the instrumental principles and techniques which allow him to analyze and measure the submicroscopic universe is not disputed, but all too often courses in chemistry take the student through fundamental work with little discussion of the application of the principles he has learned. This book fills the gap between the theoretical course work and the instruction manual of the instrument manufacturer.

The analytical techniques discussed in the text are grouped into three classes: electrochemical methods, methods involving emission or adsorption of radiation, and the methods of nuclear radiation and mass spec-

troscopy. These broad groups are subdivided into more specific areas.

The presentation of each area and its applications follows a general pattern. There is, first, a discussion of the theoretical principles which serves to refresh the memory of the chemist or student chemist. Following the discussion of principles, the applications and experimental techniques are presented and a bibliography section serves to guide further reading. Finally, a section of problems is given the student when such a section is possible, and the problems range in complexity from simple calculations to literature survey assignments. The problem section could be slightly improved if the author had included answers for at least three or four of the problems.

The author, too, has taken cognizance of the pitfalls in technique which lead to errors. These are adequately discussed and recommendations are made to minimize their occurrence. For each of the methods discussed, the author has made clear the areas in which its use may be preferred.

A final chapter serves to guide the student in laboratory experiments and, for those wishing to go further, recommendations for additional work are made.

While the book is intended for use as a text, it also should be an excellent reference book for the graduate chemist.

RICHARD H. HOLLINGER
*The Franklin Institute Laboratories*

PILOT PLANTS, MODELS, AND SCALE-UP METHODS IN CHEMICAL ENGINEERING, by Robert Edgeworth Johnstone and Meredith Wooldridge Thring. 307 pages, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1957. Price, $9.50.

The bibliography of this book contains 270 literature references classified under subject headings. The text itself is equally detailed and presents an exhaustive quantitative treatment of this subject which heretofore has received little but qualitative discussion.

It is pointed out that "there appears to be a trend toward the omission, under certain circumstances, of the pilot plant stage in the development of new processes." If the reader accepts this statement, he will ask why this book was published. The authors quickly dispel any delusions in the following terms: "For a commercial process that is required to earn profits, the decision on whether to omit the pilot plant stage or not rests upon an economic balance in which savings in time and development costs are set against higher efficiency and lower capital and start-up costs. The point of balance will vary with the growth of theoretical knowledge on the one hand and improvements in experimental methods on the other." The authors show how this trend toward more calculation and less experiment had its start and what the conclusion to this trend will probably be: ". . . more attention will be given to the elaboration and refinement of experimental techniques able to furnish quickly and at low cost design data of a higher order of accuracy than that obtainable from generalized correlations." The most important moves in this direction were the application of statistical methods to the design of experiments and the application of model theory to the scaling up (or down) of chemical plants and processes. The aim of model theory is to predict scale effects and determine the conditions under which the performance of a model gives a reliable forecast of prototype performance. The authors set themselves the task of collecting, developing, and systematizing all aspects of current model theory.

To obtain the maximum possible amount of information from a pilot plant, the "critical" components must be designed and operated in accordance with model theory. Similarity criteria which govern the operations or processes may be obtained either by dimensional analysis or from the fundamental differential equations of the process. There is a brief treatment of the factors entering design calculations, how to obtain the data, how to treat them, and where to watch for errors.

A chapter on the principle of similarity outlines some of the history and current uses of this principle with references to the literature. The authors never permit the reader to be lulled into a false sense of security by presenting their data as absolute. Without a tediously detailed treatment, they manage to warn of many of the less obvious pitfalls. The four similarity states of importance to the chemical engineer are treated with such clarity as to render understanding without

undue verbosity. The four similarity states are geometrical, mechanical, thermal, and chemical.

A brief, excellent discussion of dimensional analysis defines the exact area of utility of this principle for chemical engineers. This is followed by a chapter on differential equations which shows the information which can be derived from the latter which are known for most physical and chemical processes. The electrical analogue is utilized to show the characteristics of the information so obtainable in the form of dimensionless criteria.

The regime concept is presented as a means of distinguishing the rate-determining process which controls the over-all rate of change in a system in which several other series or parallel forces may be operating.

Chapter 7 brings together the similarity criteria for the principal types of regime that are met within chemical engineering. The requirements for similarity are specified for static, dynamic and mixed regimes where the controlling factor may be such conditions as load, mass, viscosity, gravity, surface tension, etc. Thermal and chemical regimes are also treated with their controlling factors—natural convection, radiation, mass action and surface control.

The extended principle of similarity or, extrapolation, is briefly treated. This principle permits a greater degree of flexibility because it eliminates the highly variable and incalculable shape factor. Chapter 9 discusses boundary effects or the physical and chemical effects which occur at a system's boundaries. Some methods of neutralizing or compensating for these effects are presented.

Having done an excellent job of theory presentation, the authors spend the latter two-thirds of the text demonstrating its use in detailed form for specific process equipment including ducts and flow passages, filters, heat-transfer equipment, packed towers, mixing equipment, chemical reactors, furnaces and kilns, ball mills, pressure-jet spray nozzles, centrifugal-disk atomizers, and screw extruders.

A brief chapter is devoted to corrosion and this is probably the least quantitative treatment in the book although the information will keep the engineer in mind of those corrosion factors which are most important to scale-up methods.

The final chapter, Analogue Models will be bread, meat, and potatoes to the mathematically minded.

One of the most important contributions the authors have made to this field is that of presenting a standardized terminology which appears as an appendix along with a couple of tables and a very brief discussion of instrumentation.

WILBUR S. HALL
*American Chemical Paint Company*

THE PHYSICS OF FLOW THROUGH POROUS MEDIA, by Adrian E. Scheidegger. 236 pages, diagrams, 6 × 9 in. New York, The Macmillan Company, 1957. Price, $14.00.

Although he has managed to touch on all the basic problems of porous media flow in this small book of 200 pages, the author did not intend writing a treatise or a textbook. What he did intend was to give the research worker an up-to-date, well-organized, critical guide to the literature, and this he has done very well indeed. The author writes that in compiling it he surveyed some 2000 published papers scattered throughout the journals of the Western world and Russia. The text itself includes 1300 references. He has, however, done more than merely assemble a bibliography. In logical order, he discusses the nature of porous media, the significant properties of fluids, hydrostatics, Darcy's law and permeability, and the hydrodynamics of single- and multiple-phase flow. Basic principles and current theories are well covered, with the treatment being theoretical rather than practical, and the point of view that of the physicist rather than the engineer. Nevertheless, the engineer who seeks to refine his knowledge of fluid flow (and the reviewer lists himself among these) will find this book well worthwhile.

Aside from the convenience of having all this information assembled in one place, the book's greatest merit lies in the author's careful criticism of theories and concepts. An enemy of the vague term and the fuzzy idea, he does not hesitate to make his complaints specific. Glib notions such as "pore size" and "tortuosity" are of course meaningless unless

operationally defined, but the author's illustrations indicate that many writers do not seem to realize this. Particularly welcome was his severe criticism of fundamental weaknesses in the hydraulic radius theory of permeability and of the widespread abuse of the Reynolds number. Most engineers, brought up on the hydraulic radius theory, and plagued by the variable "constants" it entails, have never heard of the more promising drag and statistical theories. Again, engineers habitually try to use the Reynolds number as a universal criterion of similitude, when in fact, as the author remarks, it can properly be so used only for straight tubes.

For all its merit, the book deserves a word or two of criticism. Dr. Scheidegger is very fussy about the terminology of others, yet he displays some remarkable eccentricities of language himself. He has a habit of using words in unfamiliar etymological senses which hinders communication and even produces mistakes (for example, "hysterical" for "hysteretic"). His repeated use of the word "heuristic" in, to judge from context, several different senses was particularly mystifying.

WILLIAM C. YAGER
*The Franklin Institute Laboratories*

PROJECT ENGINEERING OF PROCESS PLANTS, by H. F. Rase and M. H. Barrow. 692 pages, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1957. Price, $14.25.

This book is, in general, well written and can be recommended both as a text and reference for those intending to pursue careers in chemical and Petroleum Process Plant Design.

The topics discussed include all the essential phases involved in the design of a modern chemical plant ranging from the "business" end, that is, contracts, scheduling, etc., to the technical end, plant location, equipment selection, together with a resume of fundamental equations describing basic unit operations such as heat transfer and fluid flow.

This reviewer believes that the section dealing with Process Engineering could have included a more extensive treatment of preliminary plant cost estimation at the expense of reducing some of the sections dealing with a review of fundamental equations of heat transfer, etc., since it should be expected that the design engineer is already familiar with them. However, as in other chapters, sufficient references are given so that one may review the literature as the occasion arises.

One disturbing feature about this book is that it is quite expensive, which could mean that it will find its way mainly into company libraries.

The authors are experienced engineers and have spent a combined total of about 22 years in process and project work.

E. J. NOLAN
*General Electric Co.*

CHEMICAL ENGINEERING PRACTICE, VOLUME III: SOLID SYSTEMS, edited by Herbert W. Cremer and Trefor Davies. 552 pages, diagrams, 6 × 9¼ in. New York, Academic Press, Inc.; London, Butterworths Scientific Publications; 1957. Price, $17.50.

This volume of the series of twelve reviews the theory and practice of the chemical engineering operations and processes in solid systems under five broad classifications. These include: size reduction; screening, grading and classifying; mixing of solids; storage and handling of solids; and cleaning gaseous media.

In each section generally the basic principles are presented in ample detail for the student or as reference for the practicing engineer. Where space did not permit, the references and bibliography will serve adequately for the additional information. The applications of existing theories to practice are given a critical review with many illustrations of comparative data on calculated and performance results.

As the reader may anticipate, the authors have illustrated typical equipment used for the many operations both schematically as well as by photographs. The merits of the various equipment and systems are well presented. Naturally, the predominance of reference to British manufacture would be reflected by the authors' familiarity although credit is given to outstanding developments in other countries.

The economic aspects of many operations are treated mostly in general terms and justly so, since they must be analyzed on an individual basis in their respective locations. Un-

doubtedly, the editors plan a more detailed presentation of these factors in later volumes.

While the majority of the sections have been given ample space for presentation of the subjects, the author of Chapter 15 laments his space limitation. However he has limited himself in his bibliography. In such an important subject as sampling, measuring and gauging of solids with its bearing on engineering and economics, a more comprehensive treatment would be expected—at least in an up to date bibliography.

The editors again have performed an admirable task of organizing the subject matter with a negligible amount of overlapping. References are made to principles covered in other sections as well as other volumes in the series. The style of presentation is clear and easily understandable. The volume should be a valuable asset to the student of the profession and serve as a ready reference to the engineer. In addition the teacher of the subject will find its information a useful source for organizing his lectures.

R. S. DALTER
*The Franklin Institute Laboratories*

ENGINEERING ELECTRONICS WITH INDUSTRIAL APPLICATIONS, by John D. Ryder. 666 pages, diagrams, 6 × 9 in. New York, McGraw Hill Book Co., Inc., 1957. Price, $9.50.

Electronics, a science originally confined to the radio and communications art, today enjoys a widespread application to practically all fields of engineering. Many specialized engineering fields have been created as a result of efforts to apply electronic means to the solution of problems of operation and control in different areas. Numerous text books and papers have appeared dealing with these various specialties.

The electronics realm has now become so large that a real problem exists in training men for work in this field without an inordinate amount of specialization. Dr. Ryder has eased this problem somewhat by the introduction of the present text which attempts to cover in a concise manner most of the areas in which electronic methods are currently being employed.

In *Engineering Electronics*, a broad, fundamental coverage of the non-radio areas of electronics is provided. Emphasis throughout the book is on underlying principles and basic circuits applicable to various electronic devices and equipment. Recent advances in semiconductors and transistors are included—particularly a rather fundamental development of the transistor equivalent circuit from four-terminal network theory. Three chapters in the beginning provide an understanding of elementary electronics as an aid for those who might not be familiar with the theory. The latter is an illustration of the care with which the text has been put together, resulting in a clear and lucid presentation for a majority of readers. In spite of the large scope of the text, many subjects are handled in a more easily understood manner than in other books employing much greater detail.

The author states that the text is designed primarily for a senior college-level course. In the opinion of the reviewer, it excellently accomplishes this stated purpose. It could also be highly recommended to those in other engineering fields who must use electronics as a tool or aid in solving particular problems. For these, this text will serve as an excellent handbook, providing sufficient amounts of theory, and many applicable illustrations. Adequate bibliographies are provided at the end of each chapter to enable anyone whose need exceeds the scope of the text to find more detailed help. For the student, many typical problems are worked out in the text, and a considerable number is provided at the end of each chapter for practice.

There is certainly a need for a book like *Engineering Electronics* which succeeds in presenting under one cover so many of the applications for electronics that exist today. The material of the text is not new. Practically all of it can be found in greater detail in many different scattered papers. Dr. Ryder admits that the material is not complete. Volumes would be required to make it so. However, Professor Ryder should be congratulated on compiling in such excellent fashion a text, the very breadth of which is one of its prime attractions. It is clearly and intelligibly written, and is sure to be popular with all engineers whose work requires a practical, workable knowledge of the field of electronics.

DONALD L. BIRX
*The Franklin Institute Laboratories*

ELECTRICAL ENGINEERING CIRCUITS, by H. H. Skilling. 724 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1957. Price, $8.75.

This is by no means Professor Skilling's first book on electrical engineering subjects. Teaching at Stanford University since 1931, he has been a prolific contributor to the electrical literature. This reviewer had the pleasure of reviewing his "Fundamentals of Electric Waves" which was published in 1942. That volume contributed much to the teaching of microwave principles important in wartime radar technology.

The new book is a more general treatment of low frequency circuits. Intended as a teaching text rather than as a reference, the book presents an approach to all subjects which is basic and includes an adequate mathematical development for engineering purposes. To make this material more interesting, Professor Skilling has illustrated his analytical presentations with modern appliances such as the transistor. Control systems and servomechanisms are not treated, but the basic components and transform mathematics associated with them are given.

Although the content of this book includes the broad extremes of electrical engineering from communications to power, its author has again shown his unusual ability to pick out the key concepts which the student should understand. Professor Skilling is one of the rare teachers of science whose direct, precise explanation makes each subject unmistakably clear. This quality undoubtedly stems from his thorough subject knowledge as well as native teaching ability. Therefore, many of the topics which frequently receive a somewhat muddy classroom debut are presented in this volume with admirable clarity.

An excellent undergraduate course could be formulated around this text. Electrical engineering students might profit from such a course in the junior year, going on to their specialized courses in power, communications, or electronics later. Professor Skilling deserves the praise of the electrical engineering profession for his competence in preparing this work and Stanford University for tempering his duties so that its preparation was made possible.

C. W. HARGENS

*The Franklin Institute Laboratories*

GUIDED WEAPONS, by Eric Burgess. 255 pages, illustrations, 5½ × 8¼ in. New York, The Macmillan Co., 1957. Price, $5.00.

To man the development of the guided missile is second in significance only to the development of the hydrogen bomb with its technological implications. Science has reached the point where there is a threat of nuclear warheads being delivered thousands of miles from its home base using the Intercontinental Ballistic Missile and at this moment there exists no defense against the ICBM except in a negative sense. The fear of instant massive retaliation is the only deterrent known to wholesale use of guided missiles carrying nuclear warheads.

Because of these horrendous possibilities there has arisen a demand for knowledge of the operation, potentialities and types of guided missiles. While there exist many books on this subject only the latest can contain the current developments in this explosive field.

In *Guided Weapons*, Eric Burgess, a prominent figure in this field who possesses an authoritative background, has drawn a complete and lucid picture of these missiles.

The author has divided his book roughly into two sections. In the first four chapters the development and mechanics of making a guided missile are covered. Missile systems, propulsion and propellants, guidance and control and finally testing and test facilities are discussed.

The last chapter on production and development should logically be included in this section. This chapter is magnificently planned and executed and should be required reading for everyone in the missile field.

The second section contains three chapters covering the categories of guided missiles such as ground to air, air launched and finally ground to ground.

So fast have developments been cascading from the missile centers and laboratories that the most significant of these are given in less than a single page. Atlas, Titan, Thor, Jupiter C are all much in the news today with some publications giving definitive information on them. This information does not appear in *Guided Weapons*.

Despite this the author still has put to-

gether a magnificent treatment of guided missiles.

I. M. LEVITT
*The Fels Planetarium*

SOLID STATE PHYSICS, by A. J. Dekker. 540 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1957. Price, $9.00.

This volume is a useful addition to the rapidly increasing library of texts dealing with the solid state. The book was designed for senior undergraduate and beginning graduate courses. It also serves as an effective introduction to solid state physics for metallurgists, chemists, engineers and physicists already engaged in research. Since only a nominal explicit use is made of wave mechanics, extensive familiarity with this subject is not a prerequisite for understanding the subject matter. An important virtue of the book is its clarity of presentation; the subject matter is never obscured by the author's style.

The first eight chapters deal with subjects which can be presented without detailed use of electron theory. Included in these chapters are topics such as crystal structures, lattice vibration, specific heat, lattice defects, diffusion in metals and ionic crystals, lattice energies, dielectric properties, and ionic conductivity. Examples from the available data serve to illustrate the experimental techniques and physical principles involved. The reader is generally made aware of the deviations from ideal behavior found in laboratory samples of solids. This feature should be especially attractive to the neophyte experimentalist.

In the last twelve chapters of the book the electron theory of solids is developed and applied to various topics. These topics include band theory, conductivity in metals, semiconductor theory and applications, electronic properties of ionic crystals, and the magnetic properties of solids. Again, in these chapters the presentation is concise and the references to the literature are well chosen. The problems appearing at the end of the chapters are not merely computational exercises, but develop important topics in the text.

The author has succeeded admirably in achieving his purpose of presenting a well organized and clearly written introduction to the major topics in the solid state. The text will certainly serve to stimulate beginners in the field as well as those investigators who have not had a formal training in solid state theory.

A. BERKOWITZ
*The Franklin Institute Laboratories*

APPLIED PROBABILITY, by L. A. MacColl. Volume VII of the Proceedings of Symposia in Applied Mathematics. 104 pages, 7 × 10 in. New York, McGraw-Hill Book Co., Inc., 1957. Price, $5.00.

This volume is the proceedings of the Seventh Symposium in Applied Mathematics of the American Mathematical Society held at the Polytechnic Institute of Brooklyn, April 14-15, 1955. Nine papers were presented, and these are the contents of this volume. (The paper of W. Feller is in abstract form—the abstract being approximately six pages in length.)

Broadly speaking the papers fall into three general classifications and there are three papers in each category. These are:

1. The Theory of Diffusion (including the relationship between probability and differential equations):
   - *a.* "Brownian Motion Depending on $n$ Parameters: The Particular Case $n = 5$," by Paul Levy.
   - *b.* "A New Look at the First Boundary-Value Problem," by J. L. Doob.
   - *c.* "On Boundaries Defined by Stochastic Matrices," by W. Feller.

2. The Theory of Turbulence:
   - *a.* "On The Application of Functional Calculus to the Statistical Theory of Turbulence," by E. Hopf.
   - *b.* "Stochastic Processes of Astronomical Interest," by Guido Münch.
   - *c.* "The Singularity in the Spectrum of Homogeneous Turbulence," by G. K. Batchelor.

3. Probability in Classical and Modern Physics:
   - *a.* "Probability in Classical Physics," by Mark Kac.
   - *b.* "Infinite Models in Physics," by S. M. Ulam.
   - *c.* "Quantum Theory and the Foundations of Probability," by B. O. Koopman.

This volume is likely to appeal to a large class of readers. However, the particular papers that will appeal to any particular reader will depend on his tastes and interests. It is impossible, in a review of this size, to describe the contents of each paper, but this is clear, in most cases, from the description listed above. This is, probably, least true in the case of the last three papers. Hence, we include a brief description of them.

Kac's paper is concerned with equilibrium and non-equilibrium problems in physics. Under the first category he discusses problems such as the problem of the imperfect gas (concerned with the explanation of the p-v diagram and the prediction of the phenomenon of condensation on the basis of statistical mechanics). Under the second category he is concerned with the derivation of the Boltzmann equation.

The models considered by Ulam differ from conventional ones in that they start by considering an infinity of points instead of passing to the limit $n \to \infty$, and they use more general systems than the real-number system and the Euclidean continuum.

Koopman's paper deals with an apparent conflict between the classical laws of probability and quantum mechanics, and shows that this conflict can be resolved on an elementary level by making clear and explicit the concept of *event*.

H. L. PLATZER
*The Franklin Institute Laboratories*

DISPOSAL OF INDUSTRIAL WASTE MATERIALS. 157 pages, illustrations, 8½ × 11 in. London, Society of Chemical Industry; New York, The MacMillan Co., 1957. Price, $7.50.

This book contains the papers read at a Conference held at Sheffield University during the period April 17–19, 1956. The literature references at the end of each paper cover the entire field of waste water treatment and are worth while as are also the discussions which followed the presentation of each paper.

Practically every phase of waste water treatment is discussed and offers an insight into the present trend of thinking on this important subject in England. Typical of the subjects covered are "Wastes from Fine Chemicals," "Fats and Detergents," "Disposal of Colliery Wastes," "Ore Dressing Wastes and Mine Waters," "Disposal of Industrial Atomic Waste Products," etc.

The sections dealing with paper mill waste effluents are rather disappointing as contrasted with the tremendous amount of work done on this subject in this country. This deficiency is recognizable, however, since actual pulp production in England is confined mostly to straw and esparto. The paper dealing with waste material in the electroplating industry is well done and covers in detail the accepted methods for treating plating effluents and recovery of metals by ion exchange.

The pictures and diagrams accompanying the various papers are interesting and extremely helpful. The book offers an opportunity to review this important field and evaluate work done to date in what is an increasingly important aspect of modern life.

A. B. MIDDLETON
*Philadelphia Quarts Co.*

MODERN CHEMISTRY FOR THE ENGINEER AND SCIENTIST, edited by G. Ross Robertson. 442 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1957. Price, $9.50.

*Modern Chemistry for the Engineer and Scientist* consists of a series of nineteen lectures presented as an extension course at the University of California during the academic year 1954–1955. Each of the lectures, presented as individual chapters in the book, is written by a researcher who is well known for contributions in his field. The amazing variety of topics ranging from "Chemical Thermodynamics" by Kenneth S. Pitzer to "The Configuration of Polypeptide Chains in Proteins" by Linus Pauling reflects the interests of the individual authors but editor G. Ross Robertson has managed to arrange them in a logical sequence. He starts with fundamental considerations, then follows with industrial examples and concludes with lectures that show the application of chemistry to life processes.

The chapters on thermodynamics, catalysis, photochemistry, kinetics, isotopic tracers and fundamental aspects of rubber elasticity will be of great interest to a large audience for the clear manner in which the latest progress in

these fields is reported. Topics of a more specific nature include chromatography, organic reaction mechanisms, carbon fluorine compounds, silicones, the petrochemical industry, chemical relations in the earth, creep of metals, crystal chemistry and pyrochemistry of clay materials, and new developments in food technology. Among the applications of chemistry to life processes we find the biochemistry of insecticides, relations between chemical constitution and physiological action, chemical synthesis in living organisms and the paper by Pauling.

It is hard to believe that all this material is presented in 442 pages, but it has been done with clear, effective writing. It is Robertson's comment that these lectures were aimed at the graduate in physical science—perhaps a decade or more beyond his college education—for the purpose of bringing him up-to-date in a series of special chosen fields in chemistry. There is no doubt that this aim can be realized and those readers who are interested in delving deeper into particular topics will find the references cited at the end of the chapters very useful.

RAYMOND AMICONE
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## PUBLICATIONS RECEIVED

The Editors wish to call attention to these new books, received in the JOURNAL office for review. Lack of space prevents publication of more than this listing.

DATA BOOK FOR CIVIL ENGINEERS. VOLUME II. SPECIFICATIONS AND COSTS, by Elwyn E. Seelye. Revised third edition, 563 pages, 9 × 11½ in. New York, John Wiley & Sons, Inc., 1957. Price, $20.00.

CHEMISTRY, by Michell J. Sienko and Robert A. Plane. 621 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1957. Price, $6.75.

ELEMENTARY THERMODYNAMICS, by Virgil Moring Faires. Third edition, 379 pages, illustrations, 6 × 9¼ in. New York, The Macmillan Co., 1957. Price, $6.75.

PROBLEMS ON THERMODYNAMICS, by Virgil Moring Faires, Alexander V. Brewer and Clifford M. Simmang. Third edition of "Problems on Applied Thermodynamics," 165 pages, 6 × 9 in. New York, The Macmillan Co., 1957. Price, $2.50 (paper).

FUNDAMENTALS OF OPTICS, by Francis A. Jenkins and Harvey E. White. Third edition, 637 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1957. Price, $8.50.

METALLURGICAL PROGRESS—3. A third series of critical reviews, reprinted from *Iron & Steel*. 88 pages, diagrams, 8½ × 11¼ in. New York, Philosophical Library; London, Iliffe & Sons, Ltd.; n.d. Price, $6.00.

## BOOK NOTES

AIR CONDITIONING, by Willis R. Woolrich and Willis R. Woolrich, Jr. 384 pages, diagrams, 6 × 9 in. New York, The Ronald Press Co., 1957. Price, $7.50.

The authors, in preparing this text for engineering students, professional engineers and architects, have incorporated material not only on the usual problems of heating homes and commercial buildings, but also those associated with air conditioning in hot climates, both humid and dry. They cover some subjects not usually found in such texts, for example, psychrometric processes, radiant cooling and available means of refrigeration. The four parts of the book are: Thermodynamics, Psychrometrics, Load Determination; Fuels, Combustion, Heating Systems; Refrigeration, Cooling Systems, All-Year Systems; and Distribution Systems, Instrumentation, Sound Control, Filtration. Design details are omitted, for the most part, since those are available in the current trade literature.

FASTENERS HANDBOOK, by Julius Soled. 430 pages, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1957. Price, $12.50.

In searching for information on fasteners, the author discovered so little that he determined to remedy the situation. The result is this complete, up-to-date handbook covering all types of fasteners—rivets, inserts, screws, nuts, washers, retaining rings, pins, nails, metal stitching, quick release fasteners, masonry anchoring devices and hose clamps. A section is devoted to each of these over-all types, and within each section the specific kinds of fasteners are described in detail. Each kind is pictured, with its general description given. Separate paragraphs describe its special features, uses, standard materials, standard sizes. The author adds a few pertinent remarks about the product and the manufacturer's name and address are given. Each fastener described is allotted at least one page, so that all pictures of the fasteners occur at the top of the page. This accurate handbook will be invaluable to all who use fasteners of any type.

DIGITAL COMPUTERS, by R. K. Livesley. 53 pages, diagrams, 5½ × 8½ in. New York, Cambridge University Press, 1957. Price, $1.75.

One of the Cambridge Engineering Series, this book is aimed at practicing engineers and graduate students. It emphasizes the applications of computers to routine work, rather than to advanced research. There are four chapters: The Elements of Programming; Input, Storage and Output of Numbers; The Organization of Programmes; and The Solution of Engineering Problems. The purpose of the book is to acquaint the engineer with the possibilities of the use of these computers for doing calculations; therefore, no design circuits nor operational problems of the computer itself are discussed. The book should be studied by all engineers who, falsely, believe that advanced mathematical training is needed to take advantage of the computer's time-saving calculations.

THE LIVING ROCKS, photographs by Stevan Celebonovic with preface by Andre Maurois and commentary by Geoffrey Grigson. 94 pages, illustrations, 8¼ × 11¼ in. New York, Philosophical Library, n.d. Price, $6.00.

This volume, the first of the Art and Nature Series photographed and devised by Stevan Celebonovic, is a collection of outstanding photographs of minerals and fossils. Andre Maurois, in his preface, states "Surprising forms of natural beauty are revealed by the photographs of minerals and fossils in this album. Millions of years before the existence of man, there were these graceful and symmetrical lines, these regular and intricate decorative patterns—forms of beauty we tend to think of as having been invented by human artistry." For the student of photography,

the pictures speak for themselves. For the student of nature, Geoffrey Grigson has prepared a commentary on "the past in stone." And for the expert, each fossil is identified not only by name, but by location. We look forward to the remaining volumes in this unusual series.

DESIGN OF MACHINES, by Rolland T. Hinkle. 188 pages, illustrations, 6 × 9 in. New York, Prentice-Hall, Inc., 1957. Price, $4.00.

This is intended as a text for a second course in machine design, and as such, it presents an expanded view of the subject, both by introducing more advanced theories and by applying these to modern design problems. The text is so arranged that, through judicious selection of material, it can be used at several levels.

FUNDAMENTALS OF MECHANICAL DESIGN, by Richard M. Phelan. 526 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1957. Price, $8.75.

The author, an Associate Professor of Mechanical Engineering at Cornell University, has set forth the fundamentals of mechanical design in a straightforward manner. The material was originally used in a four-hour, one-semester course for electrical and agricultural students, but that original material has been amplified by the addition of chapters on castings and weldments and on power screws, to make it more generally useful.

THE TEACHING OF PHYSICS IN TROPICAL SECONDARY SCHOOLS, by H. F. Boulind. 394 pages, diagrams, 5½ × 8½ in. New York, Oxford University Press, 1957. Price, $3.40.

This is one of the UNESCO series of handbooks on the teaching of science in tropical countries, emphasizing methods of teaching rather than content. They are intended to aid the teacher to lead his pupils to an appreciation of scientific methods, in the rapidly-developing tropical countries. This present volume deals with the teaching of physics to students from 11 to 19 years of age. It covers such topics as preparation of the physics syllabus, methods of presentation, laboratory work with its corresponding safety precautions and layout, conduct of the teacher in the classroom, testing procedures, visual aids, etc.

## CURRENT TOPICS

**Automatic Floating Zone Refining.**—An automatic refining device which will reduce electrically active impurities in silicon to less than one part in one billion has been developed at Bell Telephone Laboratories. It utilizes the floating zone refining technique in which a molten zone is swept through the silicon, carrying impurities with it. The method can also be used to purify germanium, molybdenum, tungsten, and many other materials.

The new level of purity made possible by this device may lead the way to the manufacture of improved transistors, diodes and other semiconductor devices. It also will assist in completing an evaluation of the intrinsic properties of silicon and will permit a better understanding of conduction properties in high resistivity semiconductors.

Zone refining, developed at Bell Laboratories, utilizes the fact that most impurities are more soluble in molten than solid material. Thus, a molten zone swept along a rod of silicon will accumulate impurities from the silicon. The technique is being successfully applied to refining a growing number of materials: metallic, inorganic, and organic.

No satisfactory solid crucible material has been found for containing silicon during the refining process. Fused silica, frequently used in zone refining, is attacked by molten silicon. For example, contact with commercial fused silica for one hour introduces about one to three parts per billion of electrically active impurities to molten silicon at 1450° C. The floating zone process, however, requires no crucible and so is free from such contamination. In this process, the molten zone, supported only by surface tension, is passed along a vertical silicon rod rigidly held at both ends. Refining is performed in a controlled atmosphere by surrounding the rod with a water cooled gas-tight envelope. Inductive heating produces the molten zone and motion of either the heating coil or the rod sweeps the zone along. This process is repeated, with the molten zone always traversing the rod in the same direction, until the desired purity is obtained.

The new automatic apparatus developed at Bell Laboratories consists of a 10 kw, 4 mc generator, mechanical means for moving the rod up and down while it is rotating inside a gas-tight silica enclosure, and switching means for automatically recycling the mechanism.

Initially, because of its high resistivity, the end of the silicon rod must be heated to about 800° C by conduction from a molybdenum chuck which is heated by the radio frequency generator. Power is thereafter absorbed by the rod itself, and all subsequent operations are automatic. The power is increased to a preset value, sufficient to produce the desired molten zone at a preset location, and the motor drives for the mechanism are started. These drives rotate the rod and move it downward, so that the molten zone traverses the rotating rod from bottom to top. When the zone reaches the top of the rod, a limit switch stops the rotation and decreases the power to a preset value

sufficient to keep the heated but now solidified zone at about 1200° C. This heated zone is then returned to its previous position near the bottom of the rod as the rod is raised by the drive mechanism. There a limit switch raises the power to the preset value for melting, starts the rod rotating, and reverses its direction of travel. The automatic recycling continues thereafter as long as required.

Constant length and freedom from mechanical oscillation of the molten zone have been achieved by employing low heating current at a relatively high frequency and by properly proportioning the load and generator impedances. Low currents minimize electromagnetic levitation forces, the high frequency increases power transfer efficiency, and a ratio of load to generator impedance less than unity stabilizes zone length. The load impedance decreases as the rod melts so that as the zone tends to lengthen, less power is put into the load and the temperature drops, thus tending to decrease zone length. Crystals of uniform diameter are secured by control of the ratio of zone length to rod diameter so that the upward transport of material by the molten zone is balanced by the oppositely directed gravitational force.

By use of this equipment with a moist hydrogen atmosphere to remove boron, a single crystal of p-type silicon with a resistivity of 16,000 ohm cm. and minority carrier lifetime of 1200 microseconds has been obtained after 67 passes. This corresponds to an impurity content of less than one part per billion, thought to be the purest silicon ever produced in quantity.

**Spray Protects Outdoor Storage Piles of Bulk Materials.**—A new spray has been developed by The Johnson-March Corporation of Philadelphia which blankets and effectively protects outdoor storage piles of bulk materials against all types of weather conditions for periods up to a year or more.

The product, called Permaspray, is a colloidal suspension in water of a special chemical which forms a tough, flexible, water-resistant film when exposed to air and hardened. It is suitable for coating stock piles of coal, ores, coke breeze, flyash, sulphur, sand, paint pigments, waste, and all types of materials in dead storage regardless of composition or particle size.

The deposited film is said to resist the action of wind, rain, frost and heat. It also stops dust nuisance and prevents windage loss. By stretching and deforming in heat and cold the film remains unbroken. Thus it drains away water which would ordinarily seep into the stored materials, retards freezing, and reduces the normal winter damage of bulk materials stored outdoors. It also has the advantage of eliminating guttering or channeling, often caused by heavy rains, and in this way helps to prevent washing away of the pile.

Permaspray is described as being non-toxic, non-corrosive, and without adverse effect on the burning qualities of coal. When the surface film is broken, as when coal is withdrawn from the stockpile, the area can be redressed and another coating of the film applied.

Since the product has the approximate consistency of water, it can be handled easily in any type of spraying equipment with the exception of very coarse or very fine nozzles. Coarse nozzles make it difficult to control the even distribution of the spray over the pile, and nozzles that are too fine produce a mist that is too easily blown away with the wind. Permaspray

has the appearance of a white, milky, liquid and is applied without dilution.

After the film is applied, it will set in 24 hours under normal climatic conditions. The spray is applied successfully only when temperatures are above freezing. Care must be taken also to keep the solution from freezing in the drum.

The solution is supplied ready for spraying in 55-gallon steel drums, and applied at the rate of one gallon of spray to 100 square feet of surface. This mixture is said to provide effective protection for one or more years.

**New "Hi-Fi" Television Tube.**—A new type of television tube that may be the key to "hi-fi" viewing in specialized applications was exhibited by the General Electric Company in San Francisco, at the Western Electronics Show and Convention. The tube was invented several years ago by Dominic Cusano and Dr. Frank Studer of the General Electric Research Laboratory. Variations on the same principle have led to a simple two-color tube, which was also displayed.

Much finer resolution of detail is possible with the new tube, as a result of the use of an extremely thin transparent phosphor layer. This layer, which replaces the conventional coating of phosphor powder, enables the picture tube to show lines two or three times thinner than was previously possible. Ordinary phosphor powders tend to scatter the light, bouncing it from particle to particle; the new thin layer, less than one ten-thousandth of an inch thick, keeps each point of light concentrated in a small spot. Present indications point primarily to military and industrial uses.

The simple color tube called the "penetron" was developed by Dr. Lewis Koller, also of the General Electric Research Laboratory. It is made by applying transparent layers of different phosphors to the tube face. Changes in color are produced by changing the operating voltage, thereby limiting the penetration of the electron beam to the appropriate phosphor layer.

**Lightweight Electronic Trigger.**—A new type electronic trigger, compact and lightweight, has been designed and built for the U. S. Navy by Philco's Government and Industrial Division, in Philadelphia. This timing generator is unique since it provides eight separate pulse outputs in sequence, is a universal piece of equipment and uses direct power line frequency for synchronization.

Since this Philco trigger is equipped with eight separate pulse outputs, one output circuit sets up the basis pulse while the others are delayed. Thus, the unit can trigger eight operations, which may be occurring at different intervals within the same system or a different system, to occur simultaneously.

For example, a radar transmitter can be pre-started within an adjustable time interval so that the transmitted RF pulse, which picks up the target on the radar scope, will occur at the exact start of the indicator sweep to minimize error in range measurements.

Time or range measuring devices, waveform generators, video modulating devices and test equipment also can be sequentially triggered in respect to the transmitted pulse.

Being a separate piece of equipment, the Philco timing generator can be used as a control or coordinating device for individual radar units, communications and other electronic systems using 400-cycle power source.

Particularly suited to airborne operations, the trigger occupies only 0.65 cu. ft. of space and weighs 16 lb. when installed on a standard aircraft shock-mounted rack. Another outstanding feature of the Philco electronic trigger is the use of direct power line frequency, thus eliminating the separate oscillator for generating the basic timing frequency.

This new trigger is tied directly into the aircraft generator, rather than through the oscillator as has been the practice. Naturally, elimination of the oscillator reduces the weight of the new system.

Jitter on the radar scope is eliminated since the timing pulse is synchronized with the power line frequency. Jitter is much like the flutter on a television screen which is caused by a passing airplane.

Possible civilian use for the trigger could be as a universal laboratory tool for testing various equipment.

**Cleaning Semiconductor Components.**—A highly effective continuous water washing system for removing surface contamination from semiconductor components has been adapted from tube techniques by Miles V. Sullivan of Bell Telephone Laboratories. The system completely removes all the water-soluble materials which remain after etching, and monitors the effectiveness of the washing procedure.

When used on transistors, this system has resulted in significant improvements in breakdown voltage, "sharpness" of voltage-current characteristics, saturation current, and emitter reverse impedance. Thus it will contribute appreciably to the fabrication of semiconductor components having greater reliability and better operating characteristics, and will reduce the number of rejects.

Distilled water, replaced about once a week, is continuously recirculated through the system at about two liters per minute. It first passes through a small deionizing column which reduces its conductivity to about 0.1 micromho per cm as measured in a conductivity cell. The deionized water then rises through a vertical standpipe in which the parts being cleaned are suspended on a stainless steel frame. After flowing over the upper edge of the standpipe, the water passes through a second conductivity cell and thence to a sump for recirculation.

Ingenious use is made of the carbon dioxide in the air as a measuring tool. This $CO_2$ is picked up in the washing chamber, and increases the conductivity of the water in the system from 0.1 micromho per cm in the first cell to 0.15 micromho per cm in the second. Thus parts are washed until these two readings are obtained from the cells, two minutes being generally sufficient although washing is usually continued for five minutes. The thoroughness of washing is readily specified by the conductivity of the effluent and this makes possible a one-hundred per cent "inspection" of the product.

This washing technique cannot be used as a substitute for cleanliness during the entire fabrication process. Contamination must be minimized by working in air-conditioned dust-free rooms, by preventing contact with bare hands, and through use of carefully selected and highly purified materials such as cements, cutting or grinding compounds, and etching solutions.

Ultra-sonic washing techniques can be successfully employed to remove physical contaminants such as dust and lint which are not chemically bonded to the surface; and use of non-

polar organic processing materials minimizes the chemisorption or surface bonding of polar substances, thus allowing efficient removal by solvent extraction. The chemical etches customarily employed in semiconductor processing, however, produce relatively large amounts of soluble inorganic materials which are ionic and hence potentially detrimental. It is these materials which are effectively removed by the new washing system.

**New Lightweight Traveling-Wave Tube for Microwave Systems.**—A radical new traveling-wave tube that may drastically reduce the weight and substantially increase the operating reliability of microwave relay and radar equipment has been developed experimentally by the Radio Corporation of America. Its development was announced by Dr. Harwick Johnson, Acting Director, Electronic Research Laboratory, of RCA Laboratories.

The new tube, according to Dr. Johnson, "eliminates the need for bulky electromagnetic focusing equipment used with many present types of traveling-wave tubes and points the way to compact and reliable systems of new design for airborne and other applications."

The experimental tube was developed by Dr. K. K. N. Chang, of the technical staff at RCA's David Sarnoff Research Center. Dr. Chang pointed out that many traveling-wave tubes used as high-gain and high-sensitivity microwave amplifiers require external focusing electromagnets weighing thirty pounds or more, and that this focusing equipment must be so aligned that the magnetic field is precisely along the axis of the tube. Under operating conditions, he added, this alignment may be affected by vibration, changes in temperature, or other variations in environment, making realignment necessary for the effective functioning of the tube.

As described by Dr. Chang, the new experimental tube completely eliminates the external magnetic focusing equipment, substituting a compact electrostatic focusing element that is built into the tube and is automatically given a permanent proper alignment during assembly.

"The result is, in effect, a 'plug-in' traveling-wave tube which requires no critical alignment during installation and which operates with high reliability under a wide range of conditions without requiring realignment," Dr. Chang said. "Moreover, the experimental tube weighs less than a pound, complete with focusing element, as contrasted to thirty pounds or more for many tubes with the external electromagnetic focusing system."

The focusing element contained within the experimental tube, according to Dr. Chang, consists of two pairs of helices, or spiral bifilar windings. He explained that only the larger outer pair carries the radio-frequency or signal waves, while the inner pair, lying within a hollow or tubular electron beam, is used with the outer pair for beam focusing.

Although it is still in the research stage, according to Dr. Chang, the new tube has operated successfully in the laboratory as a microwave amplifier and shows promise as a new lightweight amplifier for airborne radar and counter measures equipment, as well as a receiver or amplifier for microwave relay systems.

**Thermo-Compression Bonding.**—A significant break-through in the technique of attaching leads to semiconductor devices has been achieved at Bell Telephone Laboratories. Research by O. L. Anderson, H. Christ-

ensen and P. Andreatch (at Bell Laboratories) has shown that a combination of heat and pressure can be employed to provide a firm bond between various soft metals and clean, single crystal semiconductor surfaces.

Called thermo-compression bonding, this new technique provides a bond that is stronger than the lead itself. Temperatures and pressures required are not high enough to affect the electrical properties of the semiconductor material.

One method of forming a suitable bond is to employ a heated element such as a wedge, a flat or a point, to press the metal against the heated semiconductor with a pressure sufficient to cause a slight deformation of the lead. Adhesion occurs within a matter of seconds.

Another useful connection may be made by butting the balled (or headed) end of a wire against the heated semiconductor by means of a capillary tube.

Thermo-compression bonding has a number of advantages over other methods of attaching leads to semiconductors. The bond is stronger; the technique is more readily adaptable to mass production; no chemical flux or other chemical contaminant is involved in the process; and leads may be attached to much smaller areas, an invaluable aid in fabricating high-frequency transistors.

Adhesion takes place in seconds with pressures of a few thousand pounds per square inch and temperatures well below the melting point of the alloy of the metal and semiconductor. A gold-germanium bond appears to be the easiest to make, but gold, silver, aluminum and a number of alloys can be readily bonded to either germanium or silicon.

Intensive investigations are under way to determine the mechanism of this bonding process, and to further study its practical applications. The technique is already being employed on a laboratory basis in the fabrication of transistors.

**Three-Dimensional Color TV.**—The world's first closed-circuit three-dimensional color television system has been developed by the General Electric Company for remote servicing of reactors used in development of a nuclear aircraft propulsion system.

D. Roy Shoults, general manager of the company's Aircraft Nuclear Propulsion Department (ANPD) at Evendale, Ohio, said that the color stereo system was developed to permit use of color-coded parts in reactor components and to provide the degree of precise depth perception required for their correct positioning. The new system makes remote adjustments of parts much easier than by black and white, two-dimensional TV, he said.

The new General Electric TV system was described by company engineers as "currently not feasible for the American living room," but suited for adaptation to other uses for closed-circuit television.

Development of this new system marks the first use of color television with stereo, C. Q. Lemmond, manager of particle and wave optics for the General Engineering Laboratory, said.

In use, Shoults explained, the television camera will be positioned inside the radioactive area. The viewing screen will be located a considerable distance away, behind thick shielding walls and near the controls of a mechanical manipulator. Movements of the manipulator inside the radioactive area will be directed by a technician from the 3-D color picture appearing on the screen.

The technician will view this picture

through special polarizing glasses similar to those used with recent 3-D motion pictures. Included with manipulator controls will be one used to aim and focus the twin lenses of the camera.

This system will also be used for other applications of closed-circuit television, including sales conferences, shareowner meetings, switching operations in railroad yards, process controls in automatic factories, and inspection and maintenance of work.

**Neutralizing Shower Unit.**—A portable neutralizing shower unit, designed primarily for use at tactical guided missile sites, has been developed at the Corps of Engineers' Research and Development Laboratories, Fort Belvoir, Virginia. The shower will be used by personnel who may be accidentally contaminated with liquid rocket propellants in connection with guided missile operations in all theaters.

Designed by John F. Christian, Fire Protection Engineer of the Fire Fighting Branch, this unit has a 100-gal. water tank and two air cylinders, with heating provisions, and is skid mounted. Constructed primarily of aluminum, it weighs only 570 lbs. It is 3 ft. wide, 4½ ft. long, 7½ ft. high, and can be folded up for easier transportability.

Pressurized by compressed air or nitrogen from standard gas shipping cylinders, the shower is actuated by the user by standing on a treadle. The tank is so well insulated that it requires the use of only four truck flares or highway torches to maintain the water at body temperature when the outdoor temperature is 25 degrees below zero.

An electrical immersion heating unit is available to replace the flare or open flame torches, whenever it is necessary to set up the shower unit near explosives and similar hazardous locations. One hundred feet of electric cable is supplied to connect it to the existing power system.

To protect using personnel from scalds by overheated water, a fusible type drain plug will melt out and deactivate the shower when the temperature reaches 135°F. The water is normally kept from overheating by personnel regulating the number of torches burning.

Among the other built-in safety features are a pressure relief valve and vent holes in the side of the fill cap, the latter to warn audibly of pressure within the tank when the fill cap is loosened.

An eye wash fountain is available as an accessory for installations where more extensive propellant handling operations are carried on than in tactical use.